

# Characterization of unitary processes with independent and stationary increments

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## Abstract

This is a continuation of the earlier work [13] to characterize stationary unitary increment Gaussian processes. The earlier assumption of uniform continuity is replaced by weak continuity and with a technical assumption on the domain of the generator, unitary equivalence of the processes to the solution of Hudson-Parthasarathy equation is proved.

## 1 Introduction

In [14, 15], by a co-algebraic treatment, Schürmann has proved that any weakly continuous unitary stationary independent increment process on Hilbert space  $\mathbf{h} \otimes \mathcal{H}$  ( $\mathbf{h}$  finite dimensional), is unitarily equivalent to the solution of a Hudson-Parthasarathy (HP) type quantum stochastic differential equation [7]

$$dV_t = \sum_{\mu, \nu \geq 0} V_t L_\nu^\mu \Lambda_\mu^\nu(dt), \quad V_0 = 1_{\mathbf{h} \otimes \Gamma} \quad (1.1)$$

where  $\Lambda_\mu^\nu$  are fundamental processes in the symmetric Fock space  $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$  with respect to a fixed orthonormal basis (onb) of the noise space  $\mathbf{k}$  and the coefficients  $L_\nu^\mu : \mu, \nu \geq 0$  are operators in the initial Hilbert space  $\mathbf{h}$  given by

$$L_\nu^\mu = \begin{cases} G & \text{for } (\mu, \nu) = (0, 0) \\ L_j & \text{for } (\mu, \nu) = (j, 0) \\ -\sum_{j \geq 1} L_j^* W_k^j & \text{for } (\mu, \nu) = (0, k) \\ W_k^j - \delta_k^j 1_{\mathbf{h}} & \text{for } (\mu, \nu) = (j, k) \end{cases} \quad (1.2)$$

( $\delta_k^j$  stands for Dirac delta function of  $j$  and  $k$ ) for some operators  $G, L_j$  in  $\mathbf{h}$  and a unitary operators  $W$  on  $\mathbf{h} \otimes \mathbf{k}$ .

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For characterization of Fock adapted unitary evolution see [5, 1] and references therein. In [8, 9], by extended semigroup methods, Lindsay and Wills have studied such problems for Fock adapted contractive operator cocycles and completely positive cocycles.

Recently in [13] authors have studied the case of a unitary stationary independent increment process on Hilbert space  $\mathbf{h} \otimes \mathcal{H}$  ( $\mathbf{h}$  a separable Hilbert space), with norm-continuous expectation semigroup and showed its unitary equivalent to a Hudson-Parthasarathy flow. Here we are interested in unitary processes with weakly continuous (not necessarily uniformly continuous) expectation semigroup. Under certain assumptions on the domain of the unbounded generators, extending the ideas of [13] we are able to construct the noise space  $\mathbf{k}$  and the operators (unbounded)  $G, L_j : \geq 1$  (see Proposition 4.1 and Lemma 4.3) such that the Hudson-Parthasarathy flow equation (1.1) with coefficients (1.2) (with  $W$  being identity operator), admits a unique unitary solution and the solution is unitarily equivalent to the unitary process we started with (see Theorem 5.2).

## 2 Notation and Preliminaries

We assume that all the Hilbert spaces appearing in this article are complex separable with inner product anti-linear in the first variable. For any Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$  we denote the Banach space of bounded linear operators from  $\mathcal{H}$  to  $\mathcal{K}$  and trace class operators on  $\mathcal{H}$  by  $\mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $\mathcal{B}_1(\mathcal{H})$  respectively. For a linear map (not necessarily bounded)  $T$  we write its domain as  $\mathcal{D}(T)$ . We shall denote the trace on  $\mathcal{B}_1(\mathcal{H})$  by simply  $Tr$ . The von Neumann algebra of bounded linear operators on  $\mathcal{H}$  is denoted by  $B(\mathcal{H})$ . The Banach space  $\mathcal{B}_1(\mathcal{H}, \mathcal{K}) \equiv \{\rho \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : |\rho| := \sqrt{\rho^* \rho} \in \mathcal{B}_1(\mathcal{H})\}$  with norm (Ref. Page no. 47 in [3])

$$\|\rho\|_1 = \| |\rho| \|_{\mathcal{B}_1(\mathcal{H})} = \sup \left\{ \sum_{k \geq 1} |\langle \phi_k, \rho \psi_k \rangle| : \{\phi_k\}, \{\psi_k\} \right\}$$

(  $\{\phi_k\}, \{\psi_k\}$  varies over orthonormal bases of  $\mathcal{K}$  and  $\mathcal{H}$  respectively ) is the predual of  $\mathcal{B}(\mathcal{K}, \mathcal{H})$ . For an element  $x \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ ,  $\mathcal{B}_1(\mathcal{H}, \mathcal{K}) \ni \rho \mapsto Tr(x\rho)$  defines an element of the dual Banach space  $\mathcal{B}_1(\mathcal{H}, \mathcal{K})^*$ . For a linear map  $T$  on the Banach space  $\mathcal{B}_1(\mathcal{H}, \mathcal{K})$  the adjoint  $T^*$  on the dual  $\mathcal{B}(\mathcal{K}, \mathcal{H})$  is given by  $Tr(T^*(x)\rho) := Tr(xT(\rho))$ ,  $\forall x \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ ,  $\rho \in \mathcal{B}_1(\mathcal{H}, \mathcal{K})$ .

For any  $\xi \in \mathcal{H} \otimes \mathcal{K}$ ,  $h \in \mathcal{H}$  the map

$$\mathcal{K} \ni k \mapsto \langle \xi, h \otimes k \rangle$$

defines a bounded linear functional on  $\mathcal{K}$  and thus by Riesz's theorem there exists a unique vector  $\langle\langle h, \xi \rangle\rangle$  in  $\mathcal{K}$  such that

$$\langle \langle\langle h, \xi \rangle\rangle, k \rangle = \langle \xi, h \otimes k \rangle, \forall k \in \mathcal{K}. \quad (2.1)$$

In other words  $\langle\langle h, \xi \rangle\rangle = F_h^* \xi$  where  $F_h \in \mathcal{B}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})$  is given by  $F_h k = h \otimes k$ .

Let  $\mathbf{h}$  and  $\mathcal{H}$  be two Hilbert spaces with some orthonormal bases  $\{e_j : j \geq 1\}$  and  $\{\zeta_j : j \geq 1\}$  respectively. For  $A \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  and  $u, v \in \mathbf{h}$  we define a linear operator  $A(u, v) \in \mathcal{B}(\mathcal{H})$  by

$$\langle \xi_1, A(u, v) \xi_2 \rangle = \langle u \otimes \xi_1, A v \otimes \xi_2 \rangle, \quad \forall \xi_1, \xi_2 \in \mathcal{H}$$

and read off the following properties (for a proof see Lemma 2.1 in [13]):

**Lemma 2.1.** *Let  $A, B \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  then for any  $u, v, u_i$  and  $v_i, i = 1, 2$  in  $\mathbf{h}$*

- (i)  $\|A(u, v)\| \leq \|A\| \|u\| \|v\|$  and  $A(u, v)^* = A^*(v, u)$ ,
- (ii)  $\mathbf{h} \times \mathbf{h} \mapsto A(\cdot, \cdot)$  is continuous bi-linear (anti-linear in first variable) mapping.  
If  $A(u, v) = B(u, v)$ ,  $\forall u, v \in \mathbf{h}$  then  $A = B$ ,
- (iii)  $A(u_1, v_1)B(u_2, v_2) = [A(|v_1\rangle\langle u_2| \otimes 1_{\mathcal{H}})B](u_1, v_2)$ ,
- (iv)  $AB(u, v) = \sum_{j \geq 1} A(u, e_j)B(e_j, v)$  (strongly),
- (v)  $0 \leq A(u, v)^* A(u, v) \leq \|u\|^2 A^* A(v, v)$ ,
- (vi)  $\langle A(u, v) \xi_1, B(p, w) \xi_2 \rangle = \sum_{j \geq 1} \langle p \otimes \zeta_j, [B(|w\rangle\langle v| \otimes |\xi_2\rangle\langle \xi_1|) A^* u \otimes \zeta_j] \rangle$   
 $= \langle v \otimes \xi_1, [A^* (|u\rangle\langle p| \otimes 1_{\mathcal{H}}) B w \otimes \xi_2] \rangle$ .

We also need to introduce partial trace  $Tr_{\mathcal{H}}$  which is a linear map from  $\mathcal{B}_1(\mathbf{h} \otimes \mathcal{H})$  to  $\mathcal{B}_1(\mathbf{h})$  define by, for  $B \in \mathcal{B}_1(\mathbf{h} \otimes \mathcal{H})$ ,

$$\langle u, Tr_{\mathcal{H}}(B) v \rangle := \sum_{j \geq 1} \langle u \otimes \xi_j, B v \otimes \xi_j \rangle, \quad \forall u, v \in \mathbf{h}.$$

In particular, for  $B = B_1 \otimes B_2$ ,  $Tr_{\mathcal{H}}(B) = Tr(B_2) B_1$ .

For  $A \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$ ,  $\epsilon \in \mathbb{Z}_2 = \{0, 1\}$  we define operator  $A^{(\epsilon)} \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  by  $A^{(\epsilon)} := A$  if  $\epsilon = 0$  and  $A^{(\epsilon)} := A^*$  if  $\epsilon = 1$ . For  $1 \leq k \leq n$ , we define a unitary exchange map  $P_{k,n} : \mathbf{h}^{\otimes n} \otimes \mathcal{H} \rightarrow \mathbf{h}^{\otimes n} \otimes \mathcal{H}$  by putting

$$P_{k,n}(u_1 \otimes \cdots \otimes u_n \otimes \xi) := u_1 \otimes \cdots \otimes u_{k-1} \otimes u_{k+1} \otimes \cdots \otimes u_n \otimes u_k \otimes \xi$$

on product vectors. Let  $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_2^n$ . Consider the ampliation of the operator  $A^{(\epsilon_k)}$  in  $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$  given by

$$A^{(n, \epsilon_k)} := P_{k,n}^* (1_{\mathbf{h}^{\otimes n-1}} \otimes A^{(\epsilon_k)}) P_{k,n}.$$

Now we define the operator  $A^{(\underline{\epsilon})} := \prod_{k=1}^n A^{(n, \epsilon_k)} := A^{(1, \epsilon_1)} \dots A^{(n, \epsilon_n)}$  in  $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$ . Note that as here, through out this article, the product symbol  $\prod_{k=1}^n$  stands for product with the ordering 1, 2 to  $n$ . For product vectors  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$  one can see that

$$\prod_{i=1}^m A^{(n, \epsilon_i)}(\underline{u}, \underline{v}) = \prod_{i=1}^m A^{(\epsilon_i)}(u_i, v_i) \prod_{i=m+1}^n \langle u_i, v_i \rangle \in \mathcal{B}(\mathcal{H}). \quad (2.2)$$

When  $\underline{\epsilon} = \underline{0} \in \mathbb{Z}_2^n$ , for simplicity we shall write  $A^{(n, k)}$  for  $A^{(n, \epsilon_k)}$  and  $A^{(n)}$  for  $A^{(\underline{\epsilon})}$ .

## 2.1 Symmetric Fock Space and Quantum Stochastic Calculus

Let us briefly recall the fundamental integrator processes of quantum stochastic calculus and the flow equation, introduced by Hudson and Parthasarathy [7]. For a Hilbert space  $\mathbf{k}$  let us consider the symmetric Fock space  $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ . The exponential vector in the Fock space, associated with a vector  $f \in L^2(\mathbb{R}_+, \mathbf{k})$  is given by

$$\mathbf{e}(f) = \bigoplus_{n \geq 0} \frac{1}{\sqrt{n!}} f^{(n)},$$

where  $f^{(n)} = \underbrace{f \otimes f \otimes \dots \otimes f}_{n\text{-copies}}$  for  $n > 0$  and by convention  $f^{(0)} = 1$ . The exponential vector  $\mathbf{e}(0)$  is called the vacuum vector. For any subset  $M$  of  $L^2(\mathbb{R}_+, \mathbf{k})$

we shall write  $\mathcal{E}(M)$  for the subspace spanned by  $\{\mathbf{e}(f) : f \in M\}$ . For an interval  $\Delta$  of  $\mathbb{R}_+$ , let  $\Gamma_\Delta$  be the symmetric Fock space over the Hilbert space  $L^2(\Delta, \mathbf{k}) \cong$  the range of the multiplication operator  $1_\Delta$  on  $L^2(\mathbb{R}_+, \mathbf{k})$ . For  $0 \leq s \leq t < \infty$ , the Hilbert space  $\Gamma$  decompose as  $\Gamma_s \otimes \Gamma_{(s, t]} \otimes \Gamma_{[t, \infty)}$  respectively, here we have abbreviated  $[0, s]$  by  $s]$  and  $(t, \infty)$  by  $[t$ , and for any  $f \in L^2(\mathbb{R}_+, \mathbf{k})$  the exponential vector  $\mathbf{e}(f) = \mathbf{e}(f_s] \otimes \mathbf{e}(f_{(s, t]}) \otimes \mathbf{e}(f_{[t, \infty)})$  where  $f_\Delta = 1_\Delta f$ .

Let us consider the Hudson-Parthasarathy (HP) flow equation on  $\mathbf{h} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ :

$$V_{s, t} = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_s^t V_{s, \tau} L_\nu^\mu \Lambda_\mu^\nu(d\tau). \quad (2.3)$$

Here the coefficients  $L_\nu^\mu : \mu, \nu \geq 0$  are operators in  $\mathbf{h}$  (not necessarily bounded) and  $\Lambda_\mu^\nu$  are fundamental processes with respect to a fixed orthonormal basis  $\{E_j : j \geq 1\}$  of  $\mathbf{k}$ :

$$\Lambda_\nu^\mu(t) = \begin{cases} t \, 1_{\mathbf{h} \otimes \Gamma} & \text{for } (\mu, \nu) = (0, 0) \\ a(1_{[0, t]} \otimes E_j) & \text{for } (\mu, \nu) = (j, 0) \\ a^\dagger(1_{[0, t]} \otimes E_k) & \text{for } (\mu, \nu) = (0, k) \\ \Lambda(1_{[0, t]} \otimes |E_k \rangle \langle E_j|) & \text{for } (\mu, \nu) = (j, k). \end{cases} \quad (2.4)$$

The fundamental processes  $a, a^\dagger$  and  $\Lambda$  are called annihilation, creation and conservation respectively (for their definition and detail about quantum stochastic calculus see [12, 4]).

### 3 Unitary processes with stationary and independent increments

Let  $\{U_{s,t} : 0 \leq s \leq t < \infty\}$  be a family of unitary operators in  $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  and  $\Omega$  be a fixed unit vector in  $\mathcal{H}$ . We shall write  $U_t := U_{0,t}$  for simplicity. Let us consider the family of unitary operators  $\{U_{s,t}^{(\epsilon)}\}$  in  $\mathcal{B}(\mathbf{h} \otimes \mathcal{H})$  for  $\epsilon \in \mathbb{Z}_2$  given by  $U_{s,t}^{(\epsilon)} = U_{s,t}$  if  $\epsilon = 0$ ,  $U_{s,t}^{(\epsilon)} = U_{s,t}^*$  if  $\epsilon = 1$ . As in previous section, for  $n \geq 1$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^n$  fixed and  $1 \leq k \leq n$ , we define the families of operators  $\{U_{s,t}^{(n,\epsilon_k)}\}$  and  $\{U_{s,t}^{(\underline{\epsilon})}\}$  in  $\mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$ . By identity (2.2) we have, for product vectors  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$  and  $\underline{\epsilon} \in \mathbb{Z}_2^n$ ,

$$U_{s,t}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) = \prod_{i=1}^n U_{s,t}^{(\epsilon_i)}(u_i, v_i).$$

Furthermore, for  $\underline{s} = (s_1, s_2, \dots, s_n)$ ,  $\underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq s_n \leq t_n < \infty$ , we define  $U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \mathcal{H})$  by setting

$$U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})} := \prod_{k=1}^n U_{s_k, t_k}^{(n, \epsilon_k)}. \quad (3.1)$$

Then for  $\underline{u} = \otimes_{k=1}^n u_k$ ,  $\underline{v} = \otimes_{k=1}^n v_k \in \mathbf{h}^{\otimes n}$  we have

$$U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) = \prod_{k=1}^n U_{s_k, t_k}^{(\epsilon_k)}(u_k, v_k).$$

When  $\underline{\epsilon} = \underline{0}$ , we write  $U_{\underline{s}, \underline{t}}$  for  $U_{\underline{s}, \underline{t}}^{(\underline{\epsilon})}$ . For  $\alpha, \beta \geq 0$ ,  $\underline{s} = (s_1, s_2, \dots, s_n)$ ,  $\underline{t} = (t_1, t_2, \dots, t_n)$  we write  $\alpha \leq \underline{s}, \underline{t} \leq \beta$  if  $\alpha \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq s_n \leq t_n \leq \beta$ .

We assume the following on the family of unitary  $\{U_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \mathcal{H})\}$ .

#### Assumption A

**A1 (Evolution)** For any  $0 \leq r \leq s \leq t < \infty$ ,  $U_{r,s}U_{s,t} = U_{r,t}$ .

**A2 (Independence of increments)** For any  $0 \leq s_i \leq t_i < \infty : i = 1, 2$  such that  $[s_1, t_1] \cap [s_2, t_2] = \emptyset$

(i)  $U_{s_1, t_1}(u_1, v_1)$  commutes with  $U_{s_2, t_2}(u_2, v_2)$  and  $U_{s_2, t_2}^*(u_2, v_2)$  for every  $u_i, v_i \in \mathbf{h}$ .

(ii) For  $s_1 \leq \underline{a}, \underline{b} \leq t_1$ ,  $s_2 \leq \underline{q}, \underline{r} \leq t_2$  and  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ ,  $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes m}$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^n$ ,  $\underline{\epsilon}' \in \mathbb{Z}_2^m$

$$\langle \Omega, U_{\underline{a}, \underline{b}}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) U_{\underline{q}, \underline{r}}^{(\underline{\epsilon}')}(\underline{p}, \underline{w}) \Omega \rangle = \langle \Omega, U_{\underline{a}, \underline{b}}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) \Omega \rangle \langle \Omega, U_{\underline{q}, \underline{r}}^{(\underline{\epsilon}')}(\underline{p}, \underline{w}) \Omega \rangle.$$

**A3 (Stationarity of increments)** For any  $0 \leq s \leq t < \infty$  and  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^n$

$$\langle \Omega, U_{s,t}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) \Omega \rangle = \langle \Omega, U_{t-s}^{(\underline{\epsilon})}(\underline{u}, \underline{v}) \Omega \rangle.$$

**Assumption B' (Weak / Strong continuity)**

$$\lim_{t \rightarrow 0} \langle \Omega, (U_t - 1)(u, v) \Omega \rangle = 0, \quad \forall u, v \in \mathbf{h}.$$

**Remark 3.1.** The assumption B' is an weakening of the assumption B in [13].

As in [13] we also assume the following simplifying conditions.

**Assumption C (Gaussian condition)** For any  $u_i, v_i \in \mathbf{h}$ ,

$$\epsilon_i \in \mathbb{Z}_2 : i = 1, 2, 3$$

$$\lim_{t \rightarrow 0} \frac{1}{t} \langle \Omega, (U_t^{(\epsilon_1)} - 1)(u_1, v_1) (U_t^{(\epsilon_2)} - 1)(u_2, v_2) (U_t^{(\epsilon_3)} - 1)(u_3, v_3) \Omega \rangle = 0. \quad (3.2)$$

**Assumption D (Minimality)** The set  $\mathcal{S}_0 = \{U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v}) \Omega := U_{s_1, t_1}(u_1, v_1) \cdots U_{s_n, t_n}(u_n, v_n) \Omega :$

$$\underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots \leq s_n \leq t_n <$$

$$\infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i \text{ with } u_i, v_i \in \mathbf{h}\} \text{ is total in } \mathcal{H}.$$

**Remark 3.2.** The assumption D is not really a restriction, one can as well work with replacing  $\mathcal{H}$  by span closure of  $\mathcal{S}_0$ .

**Remark 3.3.** For any dense set  $\mathcal{D} \subseteq \mathbf{h}$ ,  $\mathcal{S}_0$  will be still total if we restrict  $u_i, v_i \in \mathcal{D}$  in the assumption D.

### 3.1 Expectation Semigroups

Let us look at the various semigroups associated with the evolution  $\{U_{s,t}\}$ .

For any fixed  $n \geq 1$ , we define a family of operators  $\{T_t^{(n)}\}$  on  $\mathbf{h}^{\otimes n}$  by setting

$$\langle \phi, T_t^{(n)} \psi \rangle := \langle \Omega, U_t^{(n)}(\phi, \psi) \Omega \rangle, \quad \forall \phi, \psi \in \mathbf{h}^{\otimes n}.$$

Then in particular for product vectors  $\underline{u} = \otimes_{i=1}^n u_i$ ,  $\underline{v} = \otimes_{i=1}^n v_i \in \mathbf{h}^{\otimes n}$

$$\langle \underline{u}, T_t^{(n)} \underline{v} \rangle = \langle \Omega, U_t^{(n)}(\underline{u}, \underline{v}) \Omega \rangle = \langle \Omega, U_t(u_1, v_1) U_t(u_2, v_2) \cdots U_t(u_n, v_n) \Omega \rangle.$$

We shall write  $T_t$  for  $T_t^{(1)}$ .

**Proposition 3.4.** *Under the assumption **A** and  $B'$  the  $\{T_t^{(n)}\}$  for each  $n \geq 1$  is a strongly continuous contractive semigroup on  $\mathbf{h}^{\otimes n}$ .*

We need a Lemma for the proof of this proposition. That  $T_t^{(n)}$  is a semigroup follows exactly as in the proof of Lemma 6.1 in [13] which as well as that of following Lemma we omit.

**Lemma 3.5.** (i) *For  $1 \leq k \leq n$ ,*

$$\langle \Omega, U_t^{(n,k)}(\underline{p}, \underline{w})\Omega \rangle = \langle \underline{p}, 1_{\mathbf{h}^{(\otimes k-1)}} \otimes T_t \otimes 1_{\mathbf{h}^{(\otimes n-k)}} \underline{w} \rangle, \quad \forall \underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}. \quad (3.3)$$

*We shall denote this ampliation  $1_{\mathbf{h}^{(\otimes k-1)}} \otimes T_t \otimes 1_{\mathbf{h}^{(\otimes n-k)}}$  by  $T_t^{(n,k)}$ .*

(ii) *For any  $1 \leq m \leq n$ ,  $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$ ,*

$$\langle \Omega, \left( \prod_{k=1}^m U_t^{(n,k)} \right) (\underline{p}, \underline{w}) \Omega \rangle = \langle \underline{p}, T_t^{(m)} \otimes 1_{\mathbf{h}^{(\otimes n-m)}} \underline{w} \rangle.$$

(iii) *For any  $\phi \in \mathbf{h}^{\otimes n}$ ,*

$$\begin{aligned} & \| (U_t^{(n,k)} - 1) \phi \otimes \Omega \|^2 \\ &= \langle (1 - T_t^{(n,k)}) \phi, \phi \rangle + \langle \phi, (1 - T_t^{(n,k)}) \phi \rangle \\ &\leq 2 \| (1 - T_t) \phi \| \|\phi\|. \end{aligned}$$

(iv) *For any  $\phi \in \mathbf{h}^{\otimes n}$ ,*

$$\begin{aligned} & \| (U_t^{(n)} - 1) \phi \otimes \Omega \|^2 \\ &= \langle (1 - T_t^{(n)}) \phi, \phi \rangle + \langle \phi, (1 - T_t^{(n)}) \phi \rangle \\ &\leq 2 \| (1 - T_t^{(n)}) \phi \| \|\phi\|. \end{aligned}$$

(v) *For any  $v \in \mathbf{h}$*

$$\sum_{m \geq 1} \| (U_t - 1)(e_m, v) \Omega \|^2 = 2 \operatorname{Re} \langle v, (1 - T_t)v \rangle \leq 2 \|v\| \|(T_t - 1)v\|. \quad (3.4)$$

**Proof of the Proposition 3.4 :**

The assumption  $B'$  and definition of  $T_t$  implies that the semigroup of contractions  $\{T_t\}$  on  $\mathbf{h}$  is weakly and hence strongly continuous. To apply induction let us assume that for some  $m \geq 1$ , the contractive semigroups  $\{T_t^{(n)}\}$  are strongly

continuous for all  $1 \leq n \leq m-1$ . Now let us consider the following, for any  $\phi, \psi \in \mathbf{h}^{\otimes m}$ ,

$$\begin{aligned} & \langle \phi \otimes \Omega, (U_t^{(m)} - 1)\psi \otimes \Omega \rangle \\ &= \langle \phi \otimes \Omega, \left( \left[ \prod_{k=1}^{m-1} U_t^{(m,k)} \right] [U_t^{(m,m)}] - 1 \right) \psi \otimes \Omega \rangle \\ &= \langle \left[ \prod_{k=1}^{m-1} U_t^{(m,k)} \right]^* \phi \otimes \Omega, \left( [U_t^{(m,m)}] - 1 \right) \psi \otimes \Omega \rangle \\ &+ \langle \phi \otimes \Omega, \left( \left[ \prod_{k=1}^{m-1} U_t^{(m,k)} \right] - 1 \right) \psi \otimes \Omega \rangle. \end{aligned}$$

Taking absolute value, by Lemma 3.5 we get

$$\begin{aligned} & |\langle \phi, (T_t^{(m)} - 1_{\mathbf{h}^{\otimes m}})\psi \rangle| \\ &\leq \|\phi\| \sqrt{2} \|\psi\| \|[ (1_{\mathbf{h}^{\otimes m-1}} \otimes T_t) - 1_{\mathbf{h}^{\otimes m}} ]\psi\| + |\langle \phi, ([T_t^{(m-1)} \otimes 1_{\mathbf{h}}] - 1_{\mathbf{h}^{\otimes m}})\psi \rangle| \\ &\leq \|\phi\| \sqrt{2} \|\psi\| \|[1_{\mathbf{h}^{\otimes m-1}} \otimes (T_t - 1_{\mathbf{h}})]\psi\| + \|\phi\| \|[ (T_t^{(m-1)} - 1_{\mathbf{h}^{\otimes m-1}}) \otimes 1_{\mathbf{h}} ]\psi\|. \end{aligned}$$

So strong continuity of  $T_t^{(m-1)}$  and  $T_t$  implies  $T_t^{(m)}$  is strongly continuous.  $\square$   
Let us denote the generator of the semigroup  $T_t^{(n)}$  by  $G^{(n)}$  and for  $n=1$  by  $G$  with domain  $\mathcal{D}(G)$ .

**Lemma 3.6.** *Under the assumption **C** we have the following.*

(i) For any  $n \geq 3$ ,  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^n$

$$\lim_{t \rightarrow 0} \frac{1}{t} \langle \Omega, (U_t^{(\epsilon_1)} - 1)(u_1, v_1) \cdots (U_t^{(\epsilon_n)} - 1)(u_n, v_n) \Omega \rangle = 0. \quad (3.5)$$

(ii) For vectors  $u \in \mathbf{h}, v \in \mathcal{D}(G)$ , product vectors  $\underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$  and  $\epsilon \in \mathbb{Z}_2, \underline{\epsilon}' \in \mathbb{Z}_2^n$

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)^{(\epsilon)}(u, v) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\ &= (-1)^\epsilon \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)(u, v) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle. \end{aligned} \quad (3.6)$$

*Proof.* (i) The proof is identical to that of Lemma 6.7 in [13].

(ii) For  $\epsilon = 0$  nothing to prove. To see this for  $\epsilon = 1$  consider the following

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t + U_t^* - 2)(u, v) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\ &= - \lim_{t \rightarrow 0} \frac{1}{t} \langle [(U_t^* - 1)(U_t - 1)](u, v) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\ &= - \lim_{t \rightarrow 0} \frac{1}{t} \sum_{m \geq 1} \langle (U_t - 1)(e_m, v) \Omega, (U_t - 1)(e_m, u) (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle. \end{aligned} \quad (3.7)$$



That this limit vanishes can be seen from the following

$$\begin{aligned} & \left| \frac{1}{t} \sum_{m \geq 1} \langle (U_t - 1)(e_m, v) \Omega, (U_t - 1)(e_m, u) (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \right|^2 \\ & \leq \sum_{m \geq 1} \frac{1}{t} \|(U_t - 1)(e_m, v) \Omega\|^2 \sum_{m \geq 1} \frac{1}{t} \|(U_t - 1)(e_m, u) (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega\|^2. \end{aligned}$$

By Lemma 3.5 (v) and Lemma 2.1 (iv) the above quantity is equal to

$$\begin{aligned} & 2\operatorname{Re} \langle v, \frac{1 - T_t}{t} v \rangle \frac{1}{t} \langle (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega, [(U_t^* - 1)(U_t - 1)](u, u) (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\ & \leq 2\operatorname{Re} \langle v, \frac{1 - T_t}{t} v \rangle \frac{1}{t} \langle (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega, (2 - U_t^* - U_t)(u, u) (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \end{aligned}$$

Therefore, since  $\operatorname{Re} \langle v, \frac{1 - T_t}{t} v \rangle$  is uniformly bounded in  $t$  as  $T_t$  is strongly continuous and  $v \in \mathcal{D}(G)$ , by assumption **C** we get

$$\lim_{t \rightarrow 0} \frac{1}{t} \sum_{m \geq 1} \langle (U_t - 1)(e_m, u) \Omega, (U_t - 1)(e_m, v) (U_t^{(\epsilon')} - 1)(\underline{p}, \underline{w}) \Omega \rangle = 0.$$

Thus (3.6) follows.  $\square$

For vectors  $u, p \in \mathbf{h}$  and  $v, w \in \mathcal{D}(G)$ , the identity (3.6) gives

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)^{(\epsilon)}(u, v) \Omega, (U_t - 1)^{(\epsilon')}(p, w) \Omega \rangle \\ & = (-1)^{\epsilon + \epsilon'} \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)(u, v) \Omega, (U_t - 1)(p, w) \Omega \rangle. \end{aligned} \tag{3.8}$$

For  $m, n \geq 1$ , we define a family of operators  $\{Z_t^{(m, n)} : t \geq 0\}$  on the Banach space  $\mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$  by

$$Z_t^{(m, n)} \rho = \operatorname{Tr}_{\mathcal{H}} [U_t^{(n)} (\rho \otimes |\Omega \rangle \langle \Omega|) (U_t^{(m)})^*], \quad \rho \in \mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n}).$$

Then in particular for product vectors  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes m}, \underline{p}, \underline{w} \in \mathbf{h}^{\otimes n}$ .

$$\langle \underline{p}, Z_t^{(m, n)}(|\underline{w} \rangle \langle \underline{v}|) \underline{u} \rangle := \langle U_t^{(m)}(\underline{u}, \underline{v}) \Omega, U_t^{(n)}(\underline{p}, \underline{w}) \Omega \rangle. \tag{3.9}$$

**Lemma 3.7.** *The above family  $\{Z_t^{(m, n)}\}$  is a semigroup of contractive maps on  $\mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$ . Furthermore assumption  $B'$  implies  $\{Z_t^{(m, n)}\}$  is strongly continuous in the  $\mathcal{B}_1$  topology.*

*Proof.* For  $\rho \in \mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$

$$\begin{aligned}
\|Z_t^{(m,n)}\rho\|_1 &= \|Tr_{\mathcal{H}}[U_t^{(n)}(\rho \otimes |\Omega\rangle\langle\Omega|)(U_t^{(m)})^*]\|_1 \\
&= \sup_{\phi^{(l)} \text{ onb of } \mathbf{h}^{\otimes l} : l=m,n} \sum_{k \geq 1} |\langle \phi_k^{(n)}, Tr_{\mathcal{H}}[U_t^{(n)}(\rho \otimes |\Omega\rangle\langle\Omega|)(U_t^{(m)})^*]\phi_k^{(m)} \rangle| \\
&\leq \sup_{\phi^{(l)}} \sum_{j,k \geq 1} |\langle \phi_k^{(n)} \otimes \zeta_j, U_t^{(n)}(\rho \otimes |\Omega\rangle\langle\Omega|)(U_t^{(m)})^* \phi_k^{(m)} \otimes \zeta_j \rangle| \\
&\leq \|U_t^{(n)}(\rho \otimes |\Omega\rangle\langle\Omega|)(U_t^{(m)})^*\|_1.
\end{aligned}$$

Since for any  $l \geq 1$ ,  $\{U_t^{(l)}\}$  is a family of unitary operators

$$\|Z_t^{(m,n)}\rho\|_1 = \|\rho \otimes |\Omega\rangle\langle\Omega|\|_1 = \|\rho\|_1.$$

Proof of semigroup property of  $\{Z_t^{(m,n)}\}$  is same as in Lemma 6.4 [13]. In order to prove strong continuity  $Z_t^{(m,n)}$ , it suffices to prove the same for rank one operator  $\rho = |\underline{\mathbf{w}}\rangle\langle\underline{\mathbf{v}}|$ ,  $\underline{\mathbf{v}}, \underline{\mathbf{w}}$  product vectors in  $\mathbf{h}^{\otimes m}$  and  $\mathbf{h}^{\otimes n}$  respectively. We have

$$\begin{aligned}
&\|(Z_t^{(m,n)} - 1)(|\underline{\mathbf{w}}\rangle\langle\underline{\mathbf{v}}|)\|_1 \\
&= \sup_{\phi^{(l)} \text{ onb of } \mathbf{h}^{\otimes l} : l=m,n} \sum_{k \geq 1} |\langle \phi_k^{(n)}, (Z_t^{(m,n)} - 1)(|\underline{\mathbf{w}}\rangle\langle\underline{\mathbf{v}}|)\phi_k^{(m)} \rangle| \\
&= \sup_{\phi^{(l)}} \sum_{k \geq 1} |\langle U_t^{(m)}(\phi_k^{(m)}, \underline{\mathbf{v}})\Omega, U_t^{(n)}(\phi_k^{(n)}, \underline{\mathbf{w}})\Omega \rangle - \overline{\langle \phi_k^{(m)}, \underline{\mathbf{v}} \rangle} \langle \phi_k^{(n)}, \underline{\mathbf{w}} \rangle| \\
&\leq \sup_{\phi^{(l)}} \sum_{k \geq 1} |\langle (U_t^{(m)} - 1)(\phi_k^{(m)}, \underline{\mathbf{v}})\Omega, U_t^{(n)}(\phi_k^{(n)}, \underline{\mathbf{w}})\Omega \rangle| \\
&\quad + \sup_{\phi^{(l)}} \sum_{k \geq 1} |\overline{\langle \phi_k^{(m)}, \underline{\mathbf{v}} \rangle} \langle \Omega, (U_t^{(n)} - 1)(\phi_k^{(n)}, \underline{\mathbf{w}})\Omega \rangle| \\
&\leq \sup_{\phi^{(l)}} \left[ \sum_{k \geq 1} \|(U_t^{(m)} - 1)(\phi_k^{(m)}, \underline{\mathbf{v}})\Omega\|^2 \right]^{\frac{1}{2}} \left[ \sum_{k \geq 1} \|U_t^{(n)}(\phi_k^{(n)}, \underline{\mathbf{w}})\Omega\|^2 \right]^{\frac{1}{2}} \\
&\quad + \sup_{\phi^{(l)}} \left[ \sum_{k \geq 1} |\langle \phi_k^{(m)}, \underline{\mathbf{v}} \rangle|^2 \right]^{\frac{1}{2}} \left[ \sum_{k \geq 1} \|(U_t^{(n)} - 1)(\phi_k^{(n)}, \underline{\mathbf{w}})\Omega\|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Hence by Lemma 3.5

$$\begin{aligned}
&\|(Z_t^{(m,n)} - 1)(|\underline{\mathbf{w}}\rangle\langle\underline{\mathbf{v}}|)\|_1 \\
&\leq \|\underline{\mathbf{w}}\| \sqrt{2 \|(T_t^{(m)} - 1)\underline{\mathbf{v}}\|} + \|\underline{\mathbf{v}}\| \sqrt{2 \|(T_t^{(n)} - 1)\underline{\mathbf{w}}\|}.
\end{aligned}$$

Thus by strong continuity of the semigroup  $T_t^{(m)}$  and  $T_t^{(n)}$ , and the density of the finite rank vectors in  $\mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$  the contractive semigroup  $Z_t^{(m,n)}$  is a strongly continuous on  $\mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$ .  $\square$

We shall denote the generator of the semigroup  $Z_t^{(m,n)}$  by  $\mathcal{L}^{(m,n)}$ . For  $n \geq 1$  we shall write  $Z_t^{(n)}$  for the semigroup  $Z_t^{(n,n)}$  on the Banach space  $\mathcal{B}_1(\mathbf{h}^{\otimes n})$  with denoting its generator by  $\mathcal{L}^{(n)}$  for simplicity. Moreover, we denote the semigroup  $Z_t^{(1)}$  and its generator  $\mathcal{L}^{(1)}$  by just  $Z_t$  and  $\mathcal{L}$  respectively.

**Lemma 3.8.** *For any  $n \geq 1$ ,  $Z_t^{(n)}$  is a positive trace preserving semigroup.*

*Proof.* Positivity follows from the following, for any  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes n}$

$$\langle \underline{u}, Z_t^{(n)}(|\underline{v}\rangle\langle \underline{v}|)\underline{u} \rangle = \|U_t^{(n)}(\underline{u}, \underline{v})\Omega\|^2.$$

By definition we have

$$\begin{aligned} \text{Tr}[Z_t^{(n)}(|\underline{u}\rangle\langle \underline{v}|)] &= \sum_k \langle \underline{e}_k, Z_t^{(n)}(|\underline{u}\rangle\langle \underline{v}|)\underline{e}_k \rangle \\ &= \sum_k \langle U_t^{(n)}(\underline{e}_k, \underline{v})\Omega, U_t^{(n)}(\underline{e}_k, \underline{u})\Omega \rangle \\ &= \langle \Omega, (U_t^{(n)})^* U_t^{(n)}(\underline{v}, \underline{u})\Omega \rangle. \end{aligned}$$

Since  $U_t^{(n)}$  is unitary, we get

$$\text{Tr}[Z_t^{(n)}(|\underline{u}\rangle\langle \underline{v}|)] = \langle \underline{v}, \underline{u} \rangle = \text{Tr}(|\underline{u}\rangle\langle \underline{v}|). \quad (3.10)$$

□

Let us define a family  $\{Y_t : t \geq 0\}$  of positive contractions on  $\mathcal{B}_1(\mathbf{h})$  by  $Y_t(\rho) := T_t \rho T_t^*$ ,  $\forall \rho \in \mathcal{B}_1(\mathbf{h})$ . Since  $T_t$  is a  $C_0$ -semigroup of contraction operators on  $\mathcal{B}(\mathbf{h})$  it can be seen that  $Y_t$  is a contractive  $C_0$ -semigroup on  $\mathcal{B}_1(\mathbf{h})$ . It can also be seen that [4] the generator  $\tilde{\mathcal{L}}$  of  $Y_t$  satisfy

$$\tilde{\mathcal{L}}(\rho) = G^* \rho + \rho G, \quad \forall \rho \in \mathcal{D}_0 \equiv \{(1 - G)^{-1} \sigma (1 - G^*)^{-1} : \sigma \in \mathcal{B}_1(\mathbf{h})\}$$

and  $\mathcal{D}_0$  is a core for  $\tilde{\mathcal{L}}$ . If we define the subspace  $\mathcal{N}_0 \equiv \text{Span}\{|u\rangle\langle v|, u, v \in \mathcal{D}(G)\}$  of  $\mathcal{B}_1(\mathbf{h})$ , then it is clear that  $\mathcal{N}_0$  is dense in  $\mathcal{B}_1(\mathbf{h})$  and contained in  $\mathcal{D}_0$ .

We also need another class of semigroup. For  $m, n \geq 1$  we define a family of maps  $F_t^{(m,n)}$  on the Banach space  $\mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$  by

$$F_t^{(m,n)} \rho = \text{Tr}_{\mathcal{H}}[(U_t^{(n)})^*(\rho \otimes |\Omega\rangle\langle \Omega|)U_t^{(m)}], \quad \forall \rho \in \mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n}) \quad (3.11)$$

So in particular for product vectors  $\underline{u}, \underline{v} \in \mathbf{h}^{\otimes m}$  and  $\underline{p}, \underline{q} \in \mathbf{h}^{\otimes n}$ , we have that  $\langle \underline{p}, F_t^{(m,n)}(|\underline{q}\rangle\langle \underline{q}|)\underline{u} \rangle = \langle (U_t^{(m)})^*(\underline{u}, \underline{v})\Omega, (U_t^{(n)})^*(\underline{p}, \underline{q})\Omega \rangle$ .

**Lemma 3.9.** *For any  $m, n \geq 1$ ,  $\{F_t^{(m,n)} : t \geq 0\}$  is a strongly continuous contractive semigroup on  $\mathcal{B}_1(\mathbf{h}^{\otimes m}, \mathbf{h}^{\otimes n})$ .*

*Proof.* The proof is same as for the semigroup  $Z_t^{(m,n)}$ . □

For  $n = 1$ , we shall write  $F_t$  for the semigroup  $F_t^{(1,1)}$  on the Banach space  $\mathcal{B}_1(\mathbf{h})$  and shall denote its generator by  $\mathcal{L}'$ .

## 4 Construction of noise space

Let  $M_0 := \{(\underline{u}, \underline{v}, \underline{\epsilon}) : \underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i, u_i \in \mathbf{h}, v_i \in \mathcal{D}(G), \underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}_2^n, n \geq 1\}$  and consider the relation “ $\sim$ ” on  $M_0$  as defined in [13] :  $(\underline{u}, \underline{v}, \underline{\epsilon}) \sim (\underline{p}, \underline{w}, \underline{\epsilon}')$  if  $\underline{\epsilon} = \underline{\epsilon}'$  and  $|\underline{u} \rangle \langle \underline{v}| = |\underline{p} \rangle \langle \underline{w}| \in \mathcal{B}(\mathbf{h}^{\otimes n})$ . Expanding the vectors in term of orthonormal basis  $\{e_{\underline{j}} = e_{j_1} \otimes \dots \otimes e_{j_n} : \underline{j} = (j_1, \dots, j_n), j_1, \dots, j_n \geq 1\}$  from  $\mathcal{D}(G)$ , the identity  $|\underline{u} \rangle \langle \underline{v}| = |\underline{p} \rangle \langle \underline{w}|$  is equivalent to  $\underline{u}_{\underline{j}} \bar{v}_{\underline{k}} = \underline{p}_{\underline{j}} \bar{w}_{\underline{k}}$  for each multi-indices  $\underline{j}, \underline{k}$  which gives,  $(\underline{u}, \underline{v}, \underline{\epsilon}) \sim (\underline{p}, \underline{w}, \underline{\epsilon}') \Leftrightarrow A^{(\underline{\epsilon})}(\underline{u}, \underline{v}) = A^{(\underline{\epsilon}')}(\underline{p}, \underline{w})$  for all bounded operator  $A$  and make “ $\sim$ ” a well defined equivalence relation. Now consider the algebra  $M$  generated by  $M_0 / \sim$  with multiplication structure given by  $(\underline{u}, \underline{v}, \underline{\epsilon}) \cdot (\underline{p}, \underline{w}, \underline{\epsilon}') = (\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\epsilon} \oplus \underline{\epsilon}')$ . We define a scalar valued map  $K$  on  $M \times M$  by setting, for  $(\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0$ ,

$$K((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) := \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t^{(\underline{\epsilon})} - 1)(\underline{u}, \underline{v}) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle, \text{ if it exists.}$$

**Proposition 4.1.** *If  $\mathcal{N}_0 \subseteq \mathcal{D}(\mathcal{L})$  then we have the following.*

- (i) *The map  $K$  is a well defined positive definite kernel on  $M$ .*
- (ii) *Up to unitary equivalence there exists a unique separable Hilbert space  $\mathbf{k}$ , an embedding  $\eta : M \rightarrow \mathbf{k}$  and a representation  $\pi$  of  $M$ ,  $\pi : M \rightarrow \mathcal{B}(\mathbf{k})$  such that*

$$\{\eta(\underline{u}, \underline{v}, \underline{\epsilon}) : (\underline{u}, \underline{v}, \underline{\epsilon}) \in M_0\} \text{ is total in } \mathbf{k}, \quad (4.1)$$

$$\langle \eta(\underline{u}, \underline{v}, \underline{\epsilon}), \eta(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle = K((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) \quad (4.2)$$

and

$$\pi(\underline{u}, \underline{v}, \underline{\epsilon}) \eta(\underline{p}, \underline{w}, \underline{\epsilon}') = \eta(\underline{u} \otimes \underline{p}, \underline{v} \otimes \underline{w}, \underline{\epsilon} \oplus \underline{\epsilon}') - \langle \underline{p}, \underline{w} \rangle \eta(\underline{u}, \underline{v}, \underline{\epsilon}). \quad (4.3)$$

- (iii) *For any  $(\underline{u}, \underline{v}, \underline{\epsilon}) \in M_0$ ,  $\underline{u} = \otimes_{i=1}^n u_i, \underline{v} = \otimes_{i=1}^n v_i$  and  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$*

$$\eta(\underline{u}, \underline{v}, \underline{\epsilon}) = \sum_{i=1}^n \prod_{k \neq i} \langle u_k, v_k \rangle \eta(u_i, v_i, \epsilon_i) \quad (4.4)$$

- (iv)  $\eta(u, v, 1) = -\eta(u, v, 0), \forall u \in \mathbf{h}, v \in \mathcal{D}(G)$ .

- (v) *Writing  $\eta(u, v)$  for the vector  $\eta(u, v, 0) \in \mathbf{k}$ ,*

$$\overline{\text{Span}\{\eta(u, v) : u \in \mathbf{h}, v \in \mathcal{D}(G)\}} = \mathbf{k}. \quad (4.5)$$

*Proof.* (i) First note that for any  $(\underline{u}, \underline{v}, \underline{\epsilon}) \in M_0$ ,  $\underline{u} = \otimes_{i=1}^n u_i$ ,  $\underline{v} = \otimes_{i=1}^n v_i$ ,  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n)$  we can write

$$\begin{aligned} (U_t^{(\underline{\epsilon})} - 1)(\underline{u}, \underline{v}) &= \prod_{i=1}^n U_t^{(\epsilon_i)}(u_i, v_i) - \prod_{i=1}^n \langle u_i, v_i \rangle \\ &= \sum_{1 \leq i \leq n} (U_t - 1)^{(\epsilon_i)}(u_i, v_i) \prod_{j \neq i} \langle u_j, v_j \rangle \\ &\quad + \sum_{2 \leq l \leq n} \sum_{1 \leq i_1 < \dots < i_m \leq n} \prod_{k=1}^l (U_t - 1)^{\epsilon_{i_k}}(u_{i_k}, v_{i_k}) \prod_{j \neq i_k} \langle u_j, v_j \rangle. \end{aligned} \quad (4.6)$$

Now by Lemma 3.6, for elements  $(\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0$ ,  $\underline{\epsilon} \in \mathbb{Z}_2^m$  and  $\underline{\epsilon}' \in \mathbb{Z}_2^n$ , we have

$$\begin{aligned} K((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) &= \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t^{(\underline{\epsilon})} - 1)(\underline{u}, \underline{v}) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\ &= \sum_{1 \leq i \leq m, 1 \leq j \leq n, k \neq i} \prod_{k \neq i} \overline{\langle u_k, v_k \rangle} \prod_{l \neq j} \langle p_l, w_l \rangle \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)^{(\epsilon_i)}(u_i, v_i) \Omega, (U_t - 1)^{(\epsilon'_j)}(p_j, w_j) \Omega \rangle. \end{aligned} \quad (4.7)$$

We note that

$$\begin{aligned} &\langle (U_t - 1)(u, v) \Omega, (U_t - 1)(p, w) \Omega \rangle \\ &= \langle U_t(u, v) \Omega, U_t(p, w) \Omega \rangle - \overline{\langle u, v \rangle} \langle p, w \rangle \\ &\quad - \overline{\langle u, v \rangle} \langle \Omega, [(U_t - 1)(p, w)] \Omega \rangle \\ &\quad - \overline{\langle \Omega, [(U_t - 1)(u, v)] \Omega \rangle} \langle p, w \rangle \\ &= \langle p, (Z_t - 1)(|w \rangle \langle v|)u \rangle - \overline{\langle u, v \rangle} \langle p, [(T_t - 1)w] \rangle - \overline{\langle u, (T_t - 1)v \rangle} \langle p, w \rangle. \end{aligned}$$

Thus existence of the limits on the right hand side of (4.7) follows from the identity (3.6) since the semigroups  $T_t$  on  $\mathbf{h}$  and  $Z_t$  on  $\mathcal{B}_1(\mathbf{h})$  are strongly continuous and  $|w \rangle \langle v|$  is in  $\mathcal{D}(\mathcal{L})$ . Hence  $K$  is well defined on  $M_0$ . Now extend this to the algebra  $M$  sesqui-linearly. In particular we have

$$\begin{aligned} &K((u, v, \epsilon), (p, w, \epsilon')) \\ &= (-1)^{\epsilon + \epsilon'} \lim_{t \rightarrow 0} \left\{ \langle p, \frac{Z_t - 1}{t}(|w \rangle \langle v|)u \rangle - \overline{\langle u, v \rangle} \langle p, \frac{T_t - 1}{t}w \rangle - \overline{\langle u, \frac{T_t - 1}{t}v \rangle} \langle p, w \rangle \right\} \\ &= (-1)^{\epsilon + \epsilon'} \left\{ \langle p, \mathcal{L}(|w \rangle \langle v|)u \rangle - \overline{\langle u, v \rangle} \langle p, G w \rangle - \overline{\langle u, G v \rangle} \langle p, w \rangle \right\}. \end{aligned} \quad (4.8)$$

Positive definiteness is obvious as in [13].

(ii) The Kolmogorov's construction [12] to the pair  $(M, K)$  provides the separable Hilbert space  $\mathbf{k}$  as span closure of  $\{\eta(\underline{u}, \underline{v}, \underline{\epsilon}) : (\underline{u}, \underline{v}, \underline{\epsilon}) \in M_0\}$ . Now defining  $\pi$  by (4.3) we obtain a representation of the algebra  $M$  in  $\mathbf{k}$  (proof goes similarly as

in Lemma 7.1 [13].

(iii) For any  $(\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0$ , by (4.6) and Lemma 3.6, we have

$$\begin{aligned}
\langle \eta(\underline{u}, \underline{v}, \underline{\epsilon}), \eta(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle &= K((\underline{u}, \underline{v}, \underline{\epsilon}), (\underline{p}, \underline{w}, \underline{\epsilon}')) \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t^{(\underline{\epsilon})} - 1)(\underline{u}, \underline{v}) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\
&= \sum_{i=1}^n \prod_{k \neq i} \overline{\langle u_k, v_k \rangle} \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)^{(\epsilon_i)}(u_i, v_i) \Omega, (U_t^{(\underline{\epsilon}')} - 1)(\underline{p}, \underline{w}) \Omega \rangle \\
&= \sum_{i=1}^n \prod_{k \neq i} \overline{\langle u_k, v_k \rangle} \langle \eta(u_i, v_i, \epsilon_i), \eta(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle.
\end{aligned}$$

Since  $\{\eta(\underline{p}, \underline{w}, \underline{\epsilon}') : (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0\}$  is a total subset of  $\mathbf{k}$ , (4.4) follows.

(iv) By (3.6) we have

$$\langle \eta(u, v, 1), \eta(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle = \langle -\eta(u, v, 0), \eta(\underline{p}, \underline{w}, \underline{\epsilon}') \rangle.$$

Since  $\{\eta(\underline{p}, \underline{w}, \underline{\epsilon}') : (\underline{p}, \underline{w}, \underline{\epsilon}') \in M_0\}$  is a total subset of  $\mathbf{k}$ ,  $\eta(u, v, 1) = -\eta(u, v, 0)$ .

(v) It follows immediately from parts (iii) and (iv).  $\square$

**Remark 4.2.** The representation  $\pi$  of  $M$  in  $\mathbf{k}$  is trivial

$$\pi(\underline{u}, \underline{v}, \underline{\epsilon}) \eta(\underline{p}, \underline{w}, \underline{\epsilon}') = \langle \underline{u}, \underline{v} \rangle \eta(\underline{p}, \underline{w}, \underline{\epsilon}'). \quad (4.9)$$

If we redefine  $\mathcal{M}$  to be generated by  $\underline{u}, \underline{v} \in \mathcal{D}(G)^{\otimes n}$ , then  $\mathcal{M}$  can be a  $*$ -algebra with involution:  $(\underline{u}, \underline{v}, \underline{\epsilon})^* = (\underline{u}, \underline{v}, \underline{\epsilon}^*)$  (for notations see [13]) and it is obvious that  $\pi$  given by (4.9) is indeed a  $*$ -representation.

In the sequel, we fix an orthonormal basis  $\{E_j : j \geq 1\}$  of  $\mathbf{k}$ .

**Lemma 4.3.** Under the hypothesis of Proposition 4.1 we have the followings.

- (i) There exists a unique family of operators  $\{L_j : j \geq 1\}$  in  $\mathbf{h}$  with  $\mathcal{D}(L_j) \supseteq \mathcal{D}(G)$  such that  $\langle u, L_j v \rangle = \eta_j(u, v) := \langle E_j, \eta(u, v) \rangle, \forall u \in \mathbf{h}, v \in \mathcal{D}(G)$  and  $\sum_{j \geq 1} \|L_j v\|^2 = -2 \operatorname{Re} \langle v, G v \rangle, \forall v \in \mathcal{D}(G)$ .
- (ii) The family of operators  $\{L_j : j \geq 1\}$  satisfies  $\sum_{j \geq 1} \langle u, c_j L_j v \rangle = 0, \forall u \in \mathbf{h}, v \in \mathcal{D}(G)$  for some  $c = (c_j) \in l^2(\mathbb{N})$  implies  $c = 0$ .
- (iii) The generator  $\mathcal{L}$  of strongly continuous semigroup  $Z_t$  satisfies

$$\langle p, \mathcal{L}(|w \rangle \langle v|) u \rangle = \langle p, |Gw \rangle \langle v| u \rangle + \langle p, |w \rangle \langle Gv| u \rangle + \sum_{j \geq 1} \langle p, |L_j w \rangle \langle L_j v| u \rangle, \quad (4.10)$$

for all  $u, p \in \mathbf{h}$  and  $v, w \in \mathcal{D}(G)$ . Furthermore, the family of operators  $G, L_j : j \geq 1$  satisfies

$$\langle v, Gw \rangle + \langle Gv, w \rangle + \sum_{j \geq 1} \langle L_j v, L_j w \rangle = 0, \quad (4.11)$$

for all  $v, w \in \mathcal{D}(G)$ .

*Proof.* (i) By the identity (4.8), for any  $u \in \mathbf{h}, v \in \mathcal{D}(G)$

$$\begin{aligned} & \|\eta(u, v)\|^2 \\ &= \langle u, \mathcal{L}(|v\rangle\langle v|)u \rangle - \overline{\langle u, v \rangle} \langle u, Gv \rangle - \overline{\langle u, Gv \rangle} \langle u, v \rangle \\ &\leq \{\|\mathcal{L}(|v\rangle\langle v|)\|_1 + 2\|Gv\| \|v\|\} \|u\|^2. \end{aligned} \quad (4.12)$$

Thus the linear map  $\mathbf{h} \ni u \mapsto \eta(u, v) \in \mathbf{k}$  is a bounded linear map. Hence by Riesz's representation theorem, there exists unique linear operator  $L$  from  $\mathcal{D}(G)$  to  $\mathbf{h} \otimes \mathbf{k}$  such that  $\langle \langle u, Lv \rangle \rangle = \eta(u, v)$  where the vector  $\langle \langle u, Lv \rangle \rangle \in \mathbf{k}$  is defined as in (2.1). Equivalently, there exists a unique family of linear operator  $\{L_j : j \geq 1\}$  from  $\mathcal{D}(G)$  to  $\mathbf{h}$  such that  $Lu = \sum_{j \geq 1} L_j u \otimes E_j$  and  $\langle u, L_j v \rangle = \eta_j(u, v)$ . Now, for any  $v \in \mathcal{D}(G)$

$$\begin{aligned} \|Lv\|^2 &= \sum_j \|L_j v\|^2 = \sum_{j,k} |\eta_j(e_k, v)|^2 = \sum_k \|\eta(e_k, v)\|^2 \\ &= \sum_k \left[ \langle e_k, \mathcal{L}(|v\rangle\langle v|)e_k \rangle - \overline{\langle e_k, v \rangle} \langle e_k, Gv \rangle - \overline{\langle e_k, Gv \rangle} \langle e_k, v \rangle \right] \\ &= \text{Tr} \mathcal{L}(|v\rangle\langle v|) - \langle v, Gv \rangle - \overline{\langle v, Gv \rangle}. \end{aligned}$$

Since  $Z_t$  is trace preserving (3.10) and  $|v\rangle\langle v| \in \mathcal{D}(\mathcal{L})$  by hypothesis it follows that

$$\text{Tr} \mathcal{L}(|v\rangle\langle v|) = 0$$

and therefore

$$\|Lv\|^2 = \sum_j \|L_j v\|^2 = -\langle v, Gv \rangle - \overline{\langle v, Gv \rangle} = -2\text{Re} \langle v, Gv \rangle. \quad (4.13)$$

Note that the term on right hand side is positive since  $G$  is the generator of a contractive semigroup.

(ii) For some  $c = (c_j) \in l^2(\mathbb{N})$  let  $\langle u, \sum_{j \geq 1} c_j L_j v \rangle = 0, \forall u \in \mathbf{h}, v \in \mathcal{D}(G)$ . We have

$$0 = \langle u, \sum_{j \geq 1} c_j L_j v \rangle = \sum_{j \geq 1} c_j \langle u, L_j v \rangle = \langle \sum_{j \geq 1} \bar{c}_j E_j, \eta(u, v) \rangle.$$

Since  $\overline{\text{Span}}\{\eta(u, v) : u \in \mathbf{h}, v \in \mathcal{D}(G)\} = \mathbf{k}$ , it follows that  $\sum_{j \geq 1} \bar{c}_j E_j = 0 \in \mathbf{k}$  and hence  $c_j = 0, \forall j$ .

(iii) By part (i) and identity (4.8), for any  $u, p \in \mathbf{h}$  and  $v, w \in \mathcal{D}(G)$  we have

$$\begin{aligned} \sum_{j \geq 1} \overline{\langle u, L_j v \rangle} \langle p, L_j w \rangle &= \langle \eta(u, v), \eta(p, w) \rangle \\ &= \langle p, \mathcal{L}(|w \rangle \langle v|) u \rangle - \overline{\langle u, v \rangle} \langle p, G w \rangle - \overline{\langle u, G v \rangle} \langle p, w \rangle. \end{aligned}$$

Thus

$$\begin{aligned} &\langle p, \mathcal{L}(|w \rangle \langle v|) u \rangle \\ &= \langle p, |G w \rangle \langle v| u \rangle + \langle p, |w \rangle \langle G v| u \rangle + \sum_{j \geq 1} \langle p, |L_j w \rangle \langle L_j v| u \rangle. \end{aligned}$$

Since, for any  $v, w \in \mathcal{D}(G)$ , by identity (3.10),  $\text{Tr}[\mathcal{L}(|w \rangle \langle v|)] = 0$ , from the above identity we get

$$\langle v, G w \rangle + \langle G v, w \rangle + \sum_{j \geq 1} \langle L_j v, L_j w \rangle = 0. \quad (4.14)$$

□

**Remark 4.4.** *If there exists a positive self adjoint operator  $A$  such that  $\langle v, Av \rangle = -2\text{Re}\langle v, Gv \rangle, \forall v \in \mathcal{D}(G)$ , then  $\|Lv\|^2 = \sum_j \|L_j v\|^2 = \langle v, Av \rangle = \|A^{\frac{1}{2}}v\|^2, \forall v \in \mathcal{D}(G) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}(A^{\frac{1}{2}})$  and hence  $L$  will be closable. Closability of  $(L, \mathcal{D}(G))$  can be seen as follows. Suppose  $\{v_n\} \subseteq \mathcal{D}(G)$  converges to 0 and  $\{Lv_n\}$  is convergent. Since  $\|L(v_n - v_m)\| = \|A^{\frac{1}{2}}(v_n - v_m)\|$ , convergence of  $\{Lv_n\}$  implies  $\{A^{\frac{1}{2}}v_n\}$  is Cauchy, so convergent in  $\mathbf{h}$ . As  $A^{\frac{1}{2}}$  is a closed operator we get that  $A^{\frac{1}{2}}v_n$  converges to 0 which implies  $Lv_n$  converges to 0.*

*This can happen e.g. when  $\{T_t\}$  is a holomorphic semigroup of contractions.*

**Remark 4.5.** *If we replace  $\mathcal{D}(G)$  by any dense subset  $\mathcal{D} \subseteq \mathcal{D}(G)$ , such that  $|u \rangle \langle v| \in \mathcal{D}(\mathcal{L})$  for all  $u, v \in \mathcal{D}$ , then above Proposition 4.1 and Lemma 4.3 hold with the tensor algebra  $\mathcal{M}$  modified so as to be generated by  $(\otimes_{i=1}^n u_i, \otimes_{i=1}^n v_i) : u_i \in \mathbf{h}$  and  $v_i \in \mathcal{D}$ .*

## 5 Hudson-Parthasarathy (HP) Flows and Equivalence

In order to set up the Hudson-Parthasarathy (HP) equation and proceed further we shall work under the following extra assumption.

**Assumption E:** There exists a dense set  $\mathcal{D} \subseteq \mathcal{D}(G) \cap \mathcal{D}(G^*)$  such that  $\mathcal{D}$  is a core of  $G$  in  $\mathbf{h}$  and



**E1.**  $\mathcal{D} \subseteq \mathcal{D}(L_j^*)$  for every  $j \geq 1$ ,

**E2.**  $\mathcal{N} = \text{Span}\{|u \rangle \langle v| : u, v \in \mathcal{D}\}$  is core for the generator  $\mathcal{L}$  and  $\mathcal{L}'$  of the semigroup  $Z_t$  and  $F_t$  on  $\mathcal{B}_1(\mathbf{h})$  respectively,

**E3.**  $L_j$  maps  $\mathcal{D}$  into itself and for any  $v \in \mathcal{D}$ ,  $\sum_{j \geq 1} \|GL_j v\|^2 < \infty$ .

Since  $\mathcal{D}$  is dense in  $\mathbf{h}$  one can see, by a simple approximation argument, that  $\mathcal{N}$  is dense in  $\mathcal{B}_1(\mathbf{h})$ . Recall from the Remark 4.5 that under the assumption **E2**, replacing  $\mathcal{D}(G)$  by the core  $\mathcal{D}$  in Proposition 4.1 and Lemma 4.3, we get a separable Hilbert space  $\mathbf{k}$  generated by  $\{\eta(u, v) : u \in \mathbf{h}, v \in \mathcal{D}\}$  and linear operators  $\{L_j : j \geq 1\}$  defined on  $\mathcal{D}$ .

**Remark 5.1.** *The assumption **E1** is needed for setting up an HP equation with coefficients  $G$  and  $L_j : j \geq 1$ , assumption **E2** is to assure the existence of unique unitary HP flow. The assumption **E3** will be necessary for proving the minimality of the associated HP flow which will be needed to establish unitary equivalence of the HP flow and unitary process  $U_t$ , we started with.*

Now let us state the main result of this article.

**Theorem 5.2.** *Assume **A, B, C, D** and **E**. Then we have the following.*

(i) *The HP equation*

$$V_t = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_0^t V_r L_\nu^\mu \Lambda_\mu^\nu(dr) \quad (5.1)$$

on  $\mathcal{D} \otimes \mathcal{E}(L^2(\mathbb{R}_+, \mathbf{k}))$  with coefficients  $L_\nu^\mu$  given by

$$L_\nu^\mu = \begin{cases} G & \text{for } (\mu, \nu) = (0, 0) \\ L_j & \text{for } (\mu, \nu) = (j, 0) \\ -L_k^* & \text{for } (\mu, \nu) = (0, k) \\ 0 & \text{for } (\mu, \nu) = (j, k) \end{cases} \quad (5.2)$$

admit a unique unitary solution  $V_t$ .

(ii) *There exists a unitary isomorphism  $\tilde{\Xi} : \mathbf{h} \otimes \mathcal{H} \rightarrow \mathbf{h} \otimes \Gamma$  such that*

$$U_t = \tilde{\Xi}^* V_t \tilde{\Xi}, \quad \forall \quad t \geq 0. \quad (5.3)$$

Here we shall sketch the prove of part (i) of the Theorem and postponed the proof of (ii) to next two sub sections. In order to prove the part (i) we need the following two Lemmas. For  $\lambda > 0$ , we define the Feller set  $\beta_\lambda \subseteq \mathcal{B}(\mathbf{h})$  by

$\{x \geq 0 : \langle v, x L_0^0 w \rangle + \langle L_0^0 v, x w \rangle + \sum_{j \geq 1} \langle L_0^j v, x L_0^j w \rangle = \langle v, x G w \rangle + \langle G v, x w \rangle + \sum_{j \geq 1} \langle L_j v, x L_j w \rangle = \lambda \langle v, x w \rangle, \forall v, w \in \mathcal{D}\}$ . Similarly we define the Feller set  $\tilde{\beta}_\lambda$  for coefficients  $\tilde{L}_\nu^\mu \equiv (L_\nu^\mu)^*$ .

**Lemma 5.3.** *Under the assumption E2, the Feller Condition:  $\beta_\lambda = \{0\}$  as well as  $\widetilde{\beta}_\lambda = \{0\}$  for some  $\lambda > 0$  hold .*

*Proof.* For any  $x \geq 0$  in  $\mathcal{B}(\mathbf{h})$ ,  $v, w \in \mathcal{D}$  we have

$$\begin{aligned} \sum_{j \geq 1} \langle L_j v, x L_j w \rangle &= \langle L v, x L w \rangle = \langle x^{\frac{1}{2}} L v, x^{\frac{1}{2}} L w \rangle = \sum_{m \geq 1} \langle L v, (|x^{\frac{1}{2}} e_m \rangle \langle x^{\frac{1}{2}} e_m| \otimes 1_{\mathbf{k}}) L w \rangle \\ &= \sum_{m \geq 1} \langle \langle x^{\frac{1}{2}} e_m, L v \rangle \rangle, \langle \langle x^{\frac{1}{2}} e_m, L w \rangle \rangle \rangle = \sum_{m \geq 1} \langle \eta(x^{\frac{1}{2}} e_m, v), \eta(x^{\frac{1}{2}} e_m, w) \rangle. \end{aligned}$$

Now by (4.8)

$$\begin{aligned} \sum_{j \geq 1} \langle L_j v, x L_j w \rangle &= \sum_{m \geq 1} \langle \eta(x^{\frac{1}{2}} e_m, v), \eta(x^{\frac{1}{2}} e_m, w) \rangle \\ &= \sum_{m \geq 1} \{ \langle x^{\frac{1}{2}} e_m, \mathcal{L}(|w \rangle \langle v|) x^{\frac{1}{2}} e_m \rangle - \overline{\langle x^{\frac{1}{2}} e_m, G v \rangle} \langle x^{\frac{1}{2}} e_m, w \rangle - \overline{\langle x^{\frac{1}{2}} e_m, v \rangle} \langle x^{\frac{1}{2}} e_m, G w \rangle \} \\ &= \text{Tr}[x \mathcal{L}(|w \rangle \langle v|)] - \langle v, x G w \rangle - \langle G v, x w \rangle. \end{aligned} \tag{5.4}$$

Thus

$$\langle v, x G w \rangle + \langle G v, x w \rangle + \sum_{j \geq 1} \langle L_j v, x L_j w \rangle = \text{Tr}[x \mathcal{L}(|w \rangle \langle v|)] \tag{5.5}$$

and for any  $x \in \beta_\lambda$ ,

$$\text{Tr}[x \mathcal{L}(|w \rangle \langle v|)] = \lambda \langle v, x w \rangle = \lambda \text{Tr}(x |w \rangle \langle v|), \forall v, w \in \mathcal{D}. \tag{5.6}$$

By assumption **E2** the subspace  $\mathcal{N} = \text{Span}\{|w \rangle \langle v| : v, w \in \mathcal{D}\}$  is a core for  $\mathcal{L}$  and hence the identity (5.6) extends to  $\text{Tr}[x \mathcal{L}(\rho)] = \lambda \text{tr}(x \rho), \forall \rho \in \mathcal{D}(\mathcal{L})$ . It is also clear that for  $x \in \beta_\lambda$  the scalar map  $\phi_x : \mathcal{D}(\mathcal{L}) \ni \rho \mapsto \text{Tr}[x \mathcal{L}(\rho)] = \lambda \text{Tr}(x \rho)$  extends to a bounded linear functional on  $\mathcal{B}_1(\mathbf{h})$ . Hence  $x$  is in the domain of  $\mathcal{L}^*$  and we get

$$\begin{aligned} \text{Tr}[(|w \rangle \langle v|)(\mathcal{L}^* - \lambda)x] &= 0 \\ \Rightarrow \langle v, (\mathcal{L}^* - \lambda)x w \rangle &= 0 \\ \Rightarrow (\mathcal{L}^* - \lambda)x &= 0. \end{aligned}$$

Since  $\mathcal{L}^*$  is the generator of a  $C_0$ -semigroup  $\{Z_t^*\}$  of contraction maps on  $\mathcal{B}(\mathbf{h})$ , for  $\lambda > 0$ ,  $\mathcal{L}^* - \lambda$  is invertible and hence  $x = 0$ .

To prove  $\widetilde{\beta}_\lambda = \{0\}$  let us consider the following. By identity (3.8) for vectors

$u, p \in \mathbf{h}$  and  $v, w \in \mathcal{D}$

$$\begin{aligned}
& \langle \eta(u, v), \eta(p, w) \rangle \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t - 1)(u, v) \Omega, (U_t - 1)(p, w) \Omega \rangle \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \langle (U_t^* - 1)(u, v) \Omega, (U_t^* - 1)(p, w) \Omega \rangle \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \{ \langle U_t^*(u, v) \Omega, U_t^*(p, w) \Omega \rangle - \overline{\langle u, v \rangle} \langle p, w \rangle \\
&\quad - \overline{\langle u, v \rangle} \langle \Omega, [(U_t^* - 1)(p, w)] \Omega \rangle \\
&\quad - \langle \Omega, [(U_t^* - 1)(u, v)] \Omega \rangle \langle p, w \rangle \} \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \{ \langle p, (F_t - 1)(|w \rangle \langle v|) u \rangle - \overline{\langle u, v \rangle} \langle p, (T_t^* - 1)w \rangle - \overline{\langle u, (T_t^* - 1)v \rangle} \langle p, w \rangle \}.
\end{aligned}$$

Since by E2,  $v, w \in \mathcal{D} \subseteq \mathcal{D}(G^*)$  and  $|w \rangle \langle v| \in \mathcal{D}(\mathcal{L}')$ , we get that

$$\langle \eta(u, v), \eta(p, w) \rangle = \langle p, \mathcal{L}'(|w \rangle \langle v|) u \rangle - \overline{\langle u, v \rangle} \langle p, G^* w \rangle - \overline{\langle u, G^* v \rangle} \langle p, w \rangle. \quad (5.7)$$

Thus by (5.4) and (5.7) we have

$$\begin{aligned}
\sum_{j \geq 1} \langle L_j v, x L_j w \rangle &= \sum_{m \geq 1} \langle \eta(x^{\frac{1}{2}} e_m, v), \eta(x^{\frac{1}{2}} e_m, w) \rangle \\
&= \sum_{m \geq 1} \{ \langle x^{\frac{1}{2}} e_m, \mathcal{L}'(|w \rangle \langle v|) x^{\frac{1}{2}} e_m \rangle - \overline{\langle x^{\frac{1}{2}} e_m, v \rangle} \langle x^{\frac{1}{2}} e_m, G^* w \rangle - \overline{\langle x^{\frac{1}{2}} e_m, G^* v \rangle} \langle x^{\frac{1}{2}} e_m, w \rangle \} \\
&= \text{Tr}[x \mathcal{L}'(|w \rangle \langle v|)] - \langle G^* v, x w \rangle - \langle v, x G^* w \rangle.
\end{aligned}$$

Thus

$$\langle v, x G^* w \rangle + \langle G^* v, x w \rangle + \sum_{j \geq 1} \langle L_j v, x L_j w \rangle = \text{Tr}[x \mathcal{L}'(|w \rangle \langle v|)] \quad (5.8)$$

and for any  $x \in \widetilde{\beta_\lambda}$ ,

$$\text{Tr}[x \mathcal{L}'(|w \rangle \langle v|)] = \lambda \langle v, x w \rangle = \lambda \text{Tr}(x |w \rangle \langle v|), \forall v, w \in \mathcal{D}. \quad (5.9)$$

Since the subspace  $\mathcal{N} = \text{Span}\{|w \rangle \langle v| : v, w \in \mathcal{D}\}$  is a core for  $\mathcal{L}'$  by assumption **E2**, a similar argument as above will give that  $\widetilde{\beta_\lambda} = \{0\}$ .  $\square$

**Remark 5.4.** By (5.5) and (5.8) formally  $(\mathcal{L}' - \mathcal{L})\rho = [G^* - G, \rho], \forall \rho \in \mathcal{N}$ . Denoting the imaginary part of  $G$  by  $H$  consider the derivation  $\delta_H(\rho) = -2i[H, \rho]$ . If  $\delta_H$  is bounded then the hypothesis that the subspace  $\mathcal{N}$  is a core for  $\mathcal{L}$  implies that it is a core for  $\mathcal{L}'$  and no extra assumption is needed.

**Remark 5.5.** If  $\{T_t\}$  is a holomorphic semigroup of contractions then the hypotheses on domains of  $G^*$  and  $\mathcal{L}'$  will hold automatically.

**Lemma 5.6.** Assume the hypotheses **E1** and **E2**. For  $n \geq 1$ , setting  $L_j(n) = n L_j (n1_{\mathbf{h}} - G)^{-1}$  and  $G(n) = n^2(n1_{\mathbf{h}} - G^*)^{-1}G(n1_{\mathbf{h}} - G)^{-1}$ , we have.

(i) The operators  $L_j(n), G(n) \in \mathcal{B}(\mathbf{h})$  and  $\sum_j \|L_j(n)v\|^2 = -2 \operatorname{Re}\langle v, G(n)v \rangle$ .

(ii) For  $v \in \mathcal{D}$ ,  $\lim_{n \rightarrow \infty} L_j(n)v = L_j v$ ,  $\lim_{n \rightarrow \infty} L_j(n)^*v = L_j^*v$  and  $\lim_{n \rightarrow \infty} G(n)v = Gv$ .

*Proof.* (i) For any  $v \in \mathbf{h}$ ,

$$\begin{aligned} \sum_j \|L_j(n)v\|^2 &= \sum_j n^2 \|L_j (n1_{\mathbf{h}} - G)^{-1}v\|^2 \\ &= -2 \operatorname{Re} n^2 \langle (n1_{\mathbf{h}} - G)^{-1}v, G(n1_{\mathbf{h}} - G)^{-1}v \rangle \\ &= -2 \operatorname{Re}\langle v, G(n)v \rangle. \end{aligned}$$

(ii) Since the sequences of bounded operators  $\{nL_j(n1_{\mathbf{h}} - G)^{-1}\}$  and  $\{nL_j(n1_{\mathbf{h}} - G^*)^{-1}\}$  are uniformly norm bounded and converge strongly to identity, the requirements follows.  $\square$

### Sketch of the Proof of the part (i) of Theorem 5.2 :

For each  $n \geq 1$  we consider the family of operators,

$$L_{\nu}^{\mu}(n) = \begin{cases} G(n) = n^2(n1_{\mathbf{h}} - G^*)^{-1}G(n1_{\mathbf{h}} - G)^{-1} & \text{for } (\mu, \nu) = (0, 0) \\ L_j(n) = n L_j (n1_{\mathbf{h}} - G)^{-1} & \text{for } (\mu, \nu) = (j, 0) \\ -L_k(n)^* & \text{for } (\mu, \nu) = (0, k) \\ 0 & \text{for } (\mu, \nu) = (j, k). \end{cases} \quad (5.10)$$

By hypothesis **E1**, we have that  $\lim_{n \rightarrow \infty} L_{\nu}^{\mu}(n)v = L_{\nu}^{\mu}v, \forall v \in \mathcal{D}$  and hence there exist unique contractive solution  $\{V_t\}$  for the HP equation (5.1) (see [10, 2, 4]). To show that  $\{V_t\}$  is a isometric process we shall use the Feller condition proved in Lemma 5.3. By Proposition 3.1 in [11] (also see [10, 2]) / Theorem 7.2.3 in [4] the solution  $\{V_t\}$  of HP equation 5.1 is isometric. We shall conclude the unitarity of the process  $V_t$  by employing time reversal operator and the results in [11, 4]. As  $V_t$  satisfies the equation (5.1),  $V_t^*$  satisfies the HP equation on  $\mathcal{D} \otimes \mathcal{E}(\mathcal{K})$ , since  $\mathcal{D} \subseteq \mathcal{D}(G^*)$  by E2,

$$V_t^* = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_0^t (L_{\nu}^{\mu})^* V_r^* \Lambda_{\nu}^{\mu}(dr). \quad (5.11)$$

Let us define  $\tilde{V}_t := [1_{\mathbf{h}} \otimes \Gamma(R_t)]V_t^*[1_{\mathbf{h}} \otimes \Gamma(R_t)]$ , where  $R_t$  is the time reversal operator on  $L^2(\mathbb{R}_+, \mathbf{k})$  :

$$\begin{aligned} R_t f(x) &= f(t-x) \text{ if } x \leq t \\ &= f(x) \quad \text{if } x > t \end{aligned}$$

and  $\Gamma(A)$  denote the second quantization of operator  $A$  :  $\Gamma(A)\mathbf{e}(f) = \mathbf{e}(Af)$ . Then it can be seen that the process  $\{\tilde{V}_t\}$  satisfies the HP equation on  $\mathcal{D} \otimes \mathcal{E}(\mathcal{K})$ ,

$$\tilde{V}_t = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_0^t \tilde{V}_r \tilde{L}_\nu^\mu \Lambda_\mu^\nu(dr). \quad (5.12)$$

Since the Feller condition  $\widetilde{\beta}_\lambda = \{0\}$  for  $\tilde{L}_\nu^\mu$  holds by Lemma 5.3, the solution  $\tilde{V}_t$  and hence  $V_t^*$  is isometric or equivalently  $V_t$  is co-isometric and therefore  $V_t$  is a strongly continuous unitary process.  $\square$

**Remark 5.7.** Using identity (4.14) one construct the minimal semigroup  $\hat{Z}_t$  with generator  $\hat{\mathcal{L}}$  such that restrictions of  $\mathcal{L}$  and  $\hat{\mathcal{L}}$  to  $\mathcal{N}$  are same (see [4, 11, 10, 16]). Therefore, for any  $\lambda > 0$ , the closure  $(\lambda - \hat{\mathcal{L}})\mathcal{N} = \overline{(\lambda - \mathcal{L})\mathcal{N}} = (\lambda - \mathcal{L})\mathcal{D}(\mathcal{L})$  since by hypothesis **E2** the subspace  $\mathcal{N}$  is a core for  $\mathcal{L}$ . As  $\mathcal{L}$  is the generator of a  $C_0$ -semigroup of contractions on  $\mathcal{B}_1(\mathbf{h})$  the subspace  $(\lambda - \mathcal{L})\mathcal{D}(\mathcal{L}) = \mathcal{B}_1(\mathbf{h})$  and hence  $(\lambda - \hat{\mathcal{L}})\mathcal{N} = \mathcal{B}_1(\mathbf{h})$ . Thus by Theorem 3.2.16 (ii) and (iii) in [4] we have that  $\text{Tr}(\hat{Z}_t \rho) = \text{Tr}(\rho)$ , i.e the minimal semigroup  $\hat{Z}_t$  is conservative which also implies that the Feller condition is satisfied. We also have  $(\lambda - \hat{\mathcal{L}})\mathcal{N} = \mathcal{B}_1(\mathbf{h}) = \overline{(\lambda - \hat{\mathcal{L}})\mathcal{D}(\hat{\mathcal{L}})}$  which implies  $\mathcal{N}$  is a core for  $\hat{\mathcal{L}}$  as well and hence  $\mathcal{L} = \hat{\mathcal{L}}$ . Thus  $Z_t$  is the minimal semigroup.

For any  $0 \leq s \leq t < \infty$ , we define a unitary operator  $V_{s,t} := [1_{\mathbf{h}} \otimes \Gamma(\theta_s)]V_{t-s}[1_{\mathbf{h}} \otimes \Gamma(\theta_s^*)]$ , where  $\theta_s$  is the right shift operator on  $L^2(\mathbb{R}_+, \mathbf{k})$  :

$$\begin{aligned} \theta_s f(x) &= f(x-s) \text{ if } x \geq s \\ &= f(x) \quad \text{if } x < s. \end{aligned}$$

The adjoint of  $\theta_s$  is given by  $\theta_s^* f(x) = f(x+s)$  for all  $x \geq 0$ . We shall write the ampliation  $1_{\mathbf{h}} \otimes A$  of an operator  $A$  by same symbol  $A$  when it is clear from the

context. Since the unitary process  $V_t$  is the solution of HP equation (5.1) we have

$$\begin{aligned}
V_{s,t} &= \Gamma(\theta_s) V_{t-s} \Gamma(\theta_s^*) \\
&= 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \Gamma(\theta_s) \left\{ \int_0^{t-s} V_r L_\nu^\mu \Lambda_\mu^\nu(dr) \right\} \Gamma(\theta_s^*) \\
&= 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_0^{t-s} \Gamma(\theta_s) V_r \Gamma(\theta_s^*) L_\nu^\mu \Gamma(\theta_s) \Lambda_\mu^\nu(dr) \Gamma(\theta_s^*) \\
&= 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_s^t \Gamma(\theta_s) V_{r-s} \Gamma(\theta_s^*) L_\nu^\mu \Gamma(\theta_s) \Lambda_\mu^\nu(dr-s) \Gamma(\theta_s^*).
\end{aligned}$$

Since for any interval  $\Delta \subseteq \mathbb{R}_+$ ,  $\Gamma(\theta_s) \Lambda_\mu^\nu(\Delta-s) \Gamma(\theta_s^*) = \Lambda_\mu^\nu(\Delta)$  it follows that the unitary family  $\{V_{s,t}\}$  satisfies the HP equation

$$V_{s,t} = 1_{\mathbf{h} \otimes \Gamma} + \sum_{\mu, \nu \geq 0} \int_s^t V_{s,r} L_\nu^\mu \Lambda_\mu^\nu(dr) \quad (5.13)$$

on  $\mathcal{D} \otimes \mathcal{E}(L^2(\mathbb{R}_+, \mathbf{k}))$ . We note that  $V_t = V_{0,t}$  and  $V_{s,s} = 1_{\mathbf{h} \otimes \Gamma}$ .

As for the family of unitary operators  $\{U_{s,t}\}$  on  $\mathbf{h} \otimes \mathcal{H}$ , for  $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_2^n$  we define  $V_{s,t}^{(\underline{\epsilon})} \in \mathcal{B}(\mathbf{h}^{\otimes n} \otimes \Gamma)$  by setting  $V_{s,t}^{(\epsilon)} \in \mathcal{B}(\mathbf{h} \otimes \Gamma)$  by

$$\begin{aligned}
V_{s,t}^{(\epsilon)} &= V_{s,t} \quad \text{for } \epsilon = 0 \\
&= V_{s,t}^* \quad \text{for } \epsilon = 1.
\end{aligned}$$

The next result verifies the properties of assumption **A** for the family  $V_{s,t}$  with  $\mathbf{e}(0) \in \Gamma$  replacing  $\Omega \in \mathcal{H}$ .

**Lemma 5.8.** *The family of unitary operators  $\{V_{s,t}\}$  satisfy*

- (i) *For any  $0 \leq r \leq s \leq t < \infty$ ,  $V_{r,t} = V_{r,s} V_{s,t}$ .*
- (ii) *For  $[q,r] \cap [s,t] = \emptyset$ ,  $V_{q,r}(u,v)$  commute with  $V_{s,t}(p,w)$  and  $V_{s,t}(p,w)^*$  for every  $u, v, p, w \in \mathbf{h}$ .*
- (iii) *For any  $0 \leq s \leq t < \infty$ ,*  

$$\langle \mathbf{e}(0), V_{s,t}(u,v) \mathbf{e}(0) \rangle = \langle \mathbf{e}(0), V_{t-s}(u,v) \mathbf{e}(0) \rangle = \langle u, T_{t-s} v \rangle, \quad \forall u, v \in \mathbf{h}.$$

*Proof.* (i) For fixed  $0 \leq r \leq s \leq t < \infty$ , we set  $W_{r,t} = V_{r,s} V_{s,t}$ . Then by (5.1) we have

$$\begin{aligned}
W_{r,t} &= V_{r,s} + \sum_{\mu, \nu \geq 0} \int_s^t V_{r,s} V_{s,q} L_\nu^\mu \Lambda_\mu^\nu(dq) \\
&= W_{r,s} + \sum_{\mu, \nu \geq 0} \int_s^t W_{r,q} L_\nu^\mu \Lambda_\mu^\nu(dq).
\end{aligned}$$

Thus the family of unitary operators  $\{W_{r,t}\}$  also satisfies the HP equation (5.13). Hence by uniqueness of the solution of this quantum stochastic differential equation,  $W_{r,t} = V_{r,t}, \forall t \geq s$  and the result follows.

(ii) For any  $0 \leq s \leq t < \infty$ ,  $V_{s,t} \in \mathcal{B}(\mathbf{h} \otimes \Gamma_{[s,t]})$ . So for  $p, w \in \mathbf{h}$ ,  $V_{s,t}(p, w) \in \mathcal{B}(\Gamma_{[s,t]})$  and the statement follows.

(iii) Let us set a family of contraction operators  $\{\tilde{S}_{s,t}\}$  on  $\mathbf{h}$  by

$$\langle u, \tilde{S}_{s,t}v \rangle = \langle u \otimes \mathbf{e}(0), V_{s,t}v \otimes \mathbf{e}(0) \rangle, \quad \forall u, v \in \mathbf{h}.$$

By definition of  $V_{s,t}$ , we have  $\langle u \otimes \mathbf{e}(0), V_{s,t}v \otimes \mathbf{e}(0) \rangle = \langle u \otimes \mathbf{e}(0), \Gamma(\theta_s)V_{t-s}\Gamma(\theta_s^*)v \otimes \mathbf{e}(0) \rangle = \langle u \otimes \mathbf{e}(0), V_{0,t-s}v \otimes \mathbf{e}(0) \rangle$  and hence  $\tilde{S}_{s,t} = \tilde{S}_{0,t-s}$ . Setting  $\tilde{S}_t := \tilde{S}_{0,t}$  the family  $\{\tilde{S}_t : t \geq 0\}$  is a  $C_0$ -semigroup of contractions on  $\mathbf{h}$ . Since the unitary process  $V_{s,t}$  satisfies the HP equation (5.13), for any  $u, v \in \mathcal{D}$

$$\langle u, \tilde{S}_{s,t}v \rangle = \langle u, v \rangle + \int_s^t \langle u, \tilde{S}_{s,r}Gv \rangle dr. \quad (5.14)$$

Note that  $\mathcal{D}$  is dense core for  $G$  and  $\tilde{S}_{s,t}$  is a contractive family, so the equation (5.14) extend to  $u \in \mathbf{h}, v \in \mathcal{D}(G)$  and hence the family  $\{\tilde{S}_{s,t}\}$  satisfies the following differential equation

$$\tilde{S}_{s,t} = 1 + \int_s^t \tilde{S}_{s,r}Gdr$$

on the domain  $\mathcal{D}(G)$ . Since  $G$  is the generator of the  $C_0$ -semigroup  $\{T_t\}$  we have  $\tilde{S}_{s,t} = \tilde{S}_{t-s} = T_{t-s}$ . This proves the claim.  $\square$

Consider the family of maps  $\tilde{Z}_{s,t}$  defined by

$$\tilde{Z}_{s,t}\rho = Tr_{\mathcal{H}}[V_{s,t}(\rho \otimes |\mathbf{e}(0)\rangle\langle\mathbf{e}(0)|)V_{s,t}^*], \quad \forall \rho \in \mathcal{B}_1(\mathbf{h}).$$

As for  $Z_t$ , it can be seen that  $\tilde{Z}_{s,t}$  is a contractive family of maps on  $\mathcal{B}_1(\mathbf{h})$  and in particular, for any  $u, v, p, w \in \mathbf{h}$

$$\langle p, \tilde{Z}_{s,t}(|w\rangle\langle v|)u \rangle = \langle V_{s,t}(u, v)\mathbf{e}(0), V_{s,t}(p, w)\mathbf{e}(0) \rangle.$$

**Lemma 5.9.** *The family  $\tilde{Z}_t := \tilde{Z}_{0,t}$  is a  $C_0$ -semigroup of contraction on  $\mathcal{B}_1(\mathbf{h})$  and  $\tilde{Z}_{s,t} = \tilde{Z}_{t-s} = Z_{t-s}$ .*

*Proof.* By (5.13) and Ito's formula for  $u, v, p, w \in \mathcal{D}$

$$\begin{aligned}
& \langle p, [\tilde{Z}_{s,t} - 1](|w \rangle \langle v|) u \rangle \\
&= \langle V_{s,t}(u, v)\mathbf{e}(0), V_{s,t}(p, w)\mathbf{e}(0) \rangle - \overline{\langle u, v \rangle} \langle p, w \rangle \\
&= \int_s^t \langle V_{s,\tau}(u, v)\mathbf{e}(0), V_{s,\tau}(p, Gw)\mathbf{e}(0) \rangle d\tau + \int_s^t \langle V_{s,\tau}(u, Gv)\mathbf{e}(0), V_{s,\tau}(p, w)\mathbf{e}(0) \rangle d\tau \\
&+ \int_s^t \langle V_{s,\tau}(u, L_j v)\mathbf{e}(0), V_{s,\tau}(p, L_j w)\mathbf{e}(0) \rangle d\tau \\
&= \int_s^t \langle p, \tilde{Z}_{s,\tau}(|Gw \rangle \langle v|) u \rangle d\tau + \int_s^t \langle p, \tilde{Z}_{s,\tau}(|w \rangle \langle Gv|) u \rangle d\tau \\
&+ \sum_{j \geq 1} \int_s^t \langle p, \tilde{Z}_{s,\tau}(|L_j w \rangle \langle L_j v|) u \rangle d\tau.
\end{aligned}$$

Thus

$$\langle p, [\tilde{Z}_{s,t} - 1](\rho) u \rangle = \int_s^t \langle p, \tilde{Z}_{s,\tau} \mathcal{L}(\rho) u \rangle d\tau, \quad (5.15)$$

where  $\rho = |w \rangle \langle v|$ . Since  $\mathcal{D}$  is dense in  $\mathbf{h}$ ,  $\mathcal{N}$  is a core for  $\mathcal{L}$  and  $\tilde{Z}_{s,\tau}$  is a contractive family the equation (5.15) extends to  $u, p \in \mathbf{h}$  and  $\rho \in \mathcal{D}(\mathcal{L})$ . Thus the family  $\tilde{Z}_{s,t}$  satisfies the differential equation

$$\tilde{Z}_{s,t}(\rho) = \rho + \int_s^t \tilde{Z}_{s,\tau} \mathcal{L}(\rho) d\tau, \quad \rho \in \mathcal{D}(\mathcal{L}).$$

Since  $\mathcal{L}$  is the generator of  $C_0$ -semigroup  $Z_t$ , it follows that  $\tilde{Z}_{s,t} = \tilde{Z}_{t-s} = Z_{t-s}$ .  $\square$

## 5.1 Minimality of HP Flows

In this section we shall show the minimality of the HP flow  $V_{s,t}$  discussed above which will be needed to prove the Theorem 5.2 (ii), i.e, to establish unitary equivalence of  $U_t$  and  $V_t$ . We shall prove here that the subset  $\mathcal{S}' := \{\zeta = V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) := V_{s_1, t_1}(u_1, v_1) \cdots V_{s_n, t_n}(u_n, v_n)\mathbf{e}(0) : \underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq s_1 \leq t_1 \leq \dots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \otimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}\}$  is total in the symmetric Fock space  $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ .

Since  $\mathcal{D}$  is dense in  $\mathbf{h}$ , by Remark 3.3 the subset

$\mathcal{S} := \{\zeta = U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega := U_{s_1, t_1}(u_1, v_1) \cdots U_{s_n, t_n}(u_n, v_n)\Omega : \underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \otimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}\}$  is total in  $\mathcal{H}$ . We also note that  $\{\eta(u, v) : u \in \mathbf{h}, v \in \mathcal{D}\}$  is total in  $\mathbf{k}$ .

**Lemma 5.10.** *Under the assumption E3, for any  $v \in \mathcal{D}$ ,  $\sum_{i,j \geq 1} \|L_i L_j v\|^2 < \infty$ .*



*Proof.* For any  $j \geq 1$ ,  $L_j v \in \mathcal{D}$  and by Lemma 4.3 (i),

$$\sum_{i \geq 1} \|L_i L_j v\|^2 = -\langle L_j v, G L_j v \rangle - \langle G L_j v, L_j v \rangle.$$

Therefore

$$\sum_{i, j \geq 1} \|L_i L_j v\|^2 = -2 \operatorname{Re} \sum_{j \geq 1} \langle L_j v, G L_j v \rangle \leq 2 \left[ \sum_{j \geq 1} \|L_j v\|^2 \right]^{\frac{1}{2}} \left[ \sum_{j \geq 1} \|G L_j v\|^2 \right]^{\frac{1}{2}} < \infty.$$

□

Let  $\tau \geq 0$  be fixed. We note that for any  $0 \leq s < t \leq \tau$ ,  $u \in \mathbf{h}$ ,  $v \in \mathcal{D}$  by HP equation (5.1)

$$\begin{aligned} & \frac{1}{t-s} [V_{s,t} - 1](u, v) \mathbf{e}(0) \\ &= \frac{1}{t-s} \left\{ \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j v) a_j^\dagger(d\lambda) + \int_s^t V_{s,\lambda}(u, Gv) d\lambda \right\} \mathbf{e}(0) \\ &= \gamma(s, t, u, v) + \langle u, Gv \rangle \mathbf{e}(0) + \zeta(s, t, u, v) + \varsigma(s, t, u, v) \end{aligned} \quad (5.16)$$

where these vectors in the Fock space  $\Gamma$  are given by

$$\begin{aligned} \gamma(s, t, u, v) &:= \frac{1}{t-s} \sum_{j \geq 1} \langle u, L_j v \rangle a_j^\dagger([s, t]) \mathbf{e}(0) \\ \zeta(s, t, u, v) &:= \frac{1}{t-s} \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1)(u, L_j v) a_j^\dagger(d\lambda) \mathbf{e}(0) \\ \varsigma(s, t, u, v) &:= \frac{1}{t-s} \int_s^t (V_{s,\lambda} - 1)(u, Gv) d\lambda \mathbf{e}(0). \end{aligned}$$

Note that any  $\xi \in \Gamma$  can be written as  $\xi = \xi^{(0)} \mathbf{e}(0) \oplus \xi^{(1)} \oplus \dots \oplus \xi^{(n)}$  in the  $n$ -fold symmetric tensor product  $L^2(\mathbb{R}_+, \mathbf{k})^{\otimes n} \equiv L^2(\Sigma_n) \otimes \mathbf{k}^{\otimes n}$  where  $\Sigma_n$  is the  $n$ -simplex  $\{\underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq t_1 < t_2 < \dots < t_n < \infty\}$ .

**Lemma 5.11.** *For any  $u \in \mathbf{h}$ ,  $v \in \mathcal{D}$ ,  $0 \leq s \leq t \leq \tau$*

$$\left\| \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j v) a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2 \leq C_\tau (t-s) \|u\|^2 \sum_{j \geq 1} \|L_j v\|^2 \quad (5.17)$$

where  $C_\tau = 2e^\tau$

*Proof.* For any  $\phi$  in the Fock space  $\Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ ,

$$\begin{aligned} & \left| \langle \phi, \sum_{j \geq 1} \int_s^t V_{s,\lambda}(u, L_j v) a_j^\dagger(d\lambda) \mathbf{e}(0) \rangle \right|^2 \\ &= \left| \langle u \otimes \phi, \left\{ \sum_{j \geq 1} \int_s^t V_{s,\lambda} a_j^\dagger(d\lambda) \right\} L_j v \otimes \mathbf{e}(0) \rangle \right|^2 \\ &\leq \|u \otimes \phi\|^2 \left\| \left\{ \sum_{j \geq 1} \int_s^t V_{s,\lambda} a_j^\dagger(d\lambda) \right\} L_j v \otimes \mathbf{e}(0) \right\|^2. \end{aligned}$$

By estimate of quantum stochastic integration (Proposition 27.1, [12]), the above quantity is

$$\begin{aligned} &\leq C_\tau \|u \otimes \phi\|^2 \sum_{j \geq 1} \int_s^t \|V_{s,\lambda} L_j v \otimes \mathbf{e}(0)\|^2 d\lambda \\ &\leq C_\tau (t-s) \|u \otimes \phi\|^2 \sum_{j \geq 1} \|L_j v\|^2. \end{aligned}$$

Since  $\phi$  is arbitrary requirement follows. □

**Lemma 5.12.** *For any  $u \in \mathbf{h}, v \in \mathcal{D}, 0 \leq s \leq t \leq \tau$  there exist constants  $C_{\tau,u,v}, C'_{\tau,u,v}$  given by*

$$C_{\tau,u,v} = 2\|u\|^2 [C_\tau \sum_{j \geq 1} \|L_j v\|^2 + \tau \|G v\|^2]$$

and

$$C'_{\tau,u,v} = 2C_\tau \|u\|^2 [C_\tau \sum_{i,j \geq 1} \|L_j L_i v\|^2 + \tau \sum_{i \geq 1} \|G L_i v\|^2]$$

such that

- (i)  $\|(V_{s,t} - 1)(u, v) \otimes \mathbf{e}(0)\|^2 \leq C_{\tau,u,v}(t-s)$
- (ii)  $\|\zeta(s, t, u, v)\|^2 \leq C'_{\tau,u,v}$  and  $\|\zeta(s, t, u, v)\| \leq C_{\tau,u,v} \sqrt{t-s}, \forall 0 \leq s < t \leq \tau.$
- (iii) For any  $\xi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$ ,  $\lim_{s \rightarrow t} \langle \xi, \zeta(s, t, u, v) \rangle = 0$  and

$$\lim_{s \rightarrow t} \langle \xi, \gamma(s, t, u, v) \rangle = \sum_{j \geq 1} \langle u, L_j v \rangle \overline{\xi_j^{(1)}}(t) = \langle \xi^{(1)}(t), \eta(u, v) \rangle, \quad a.e. \quad t \geq 0.$$

*Proof.* (i) By identity (5.16) and Lemma 5.11 we have

$$\begin{aligned} &\|(V_{s,t} - 1)(u, v) \otimes \mathbf{e}(0)\|^2 \\ &= \left\| \sum_{j \geq 1} \int_s^t V_{s,\alpha}(u, L_j v) a_j^\dagger(d\alpha) \otimes \mathbf{e}(0) + \int_s^t V_{s,\alpha}(u, Gv) \otimes \mathbf{e}(0) d\alpha \right\|^2 \\ &\leq 2 \left\| \sum_{j \geq 1} \int_s^t V_{s,\alpha}(u, L_j v) d\alpha \otimes \mathbf{e}(0) \right\|^2 + \left[ \int_s^t \|V_{s,\alpha}(u, Gv) \otimes \mathbf{e}(0)\| d\alpha \right]^2 \\ &\leq 2\|u\|^2 [C_\tau(t-s) \sum_{j \geq 1} \|L_j v\|^2 + [(t-s)\|G v\|]^2] \\ &\leq C_{\tau,u,v}(t-s). \end{aligned}$$

(ii) 1. As in the proof of Lemma 5.11 we have

$$\begin{aligned}\|\zeta(s, t, u, v)\|^2 &= \frac{1}{(t-s)^2} \left\| \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1)(u, L_j v) a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2 \\ &\leq \frac{\|u\|^2}{(t-s)^2} \left\| \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1) L_j v a_j^\dagger(d\lambda) \mathbf{e}(0) \right\|^2.\end{aligned}$$

Since  $L_j v \in \mathcal{D}$  for all  $j \geq 1$  by assumption **E3**, by estimate of quantum stochastic integration (Proposition 27.1, [12]) the above quantity is

$$\begin{aligned}&\leq \frac{C_\tau \|u\|^2}{(t-s)^2} \sum_{j \geq 1} \int_s^t \|(V_{s,\lambda} - 1) L_j v \mathbf{e}(0)\|^2 d\lambda \\ &\leq 2 \frac{C_\tau \|u\|^2}{(t-s)^2} \sum_{j \geq 1} (t-s) [C_\tau(t-s) \sum_{i \geq 1} \|L_i L_j v\|^2 + (t-s)^2 \|G L_j v\|^2] \\ &\leq 2 C_\tau \|u\|^2 \sum_{j \geq 1} [C_\tau \sum_{i \geq 1} \|L_i L_j v\|^2 + (t-s) \|G L_j v\|^2] \\ &\leq 2 C_\tau \|u\|^2 \sum_{i \geq 1} [C_\tau \sum_{j \geq 1} \|L_j L_i v\|^2 + \tau \|G L_i v\|^2] = C'_{\tau, u, v}\end{aligned}$$

2. We have

$$\begin{aligned}\|\zeta(s, t, u, v)\| &= \frac{1}{(t-s)} \left\| \int_s^t (V_{s,\lambda} - 1)(u, Gv) d\lambda \mathbf{e}(0) \right\| \\ &\leq \frac{1}{(t-s)} \int_s^t \|(V_{s,\lambda} - 1)(u, Gv) \mathbf{e}(0)\| d\lambda.\end{aligned}$$

By part (i) it follows that  $\|\zeta(s, t, u, v)\|^2 \leq C_{\tau, u, v} \sqrt{t-s}$ .

(iii) 1. For any  $f \in L^2(\mathbb{R}_+, \mathbf{k})$  let us consider

$$\begin{aligned}\langle \mathbf{e}(f), \zeta(s, t, u, v) \rangle &= \langle \mathbf{e}(f), \frac{1}{t-s} \sum_{j \geq 1} \int_s^t (V_{s,\lambda} - 1)(u, L_j v) a_j^\dagger(d\lambda) \mathbf{e}(0) \rangle \\ &= \frac{1}{t-s} \sum_{j \geq 1} \int_s^t \overline{f_j(\lambda)} \langle \mathbf{e}(f), (V_{s,\lambda} - 1)(u, L_j v) \mathbf{e}(0) \rangle d\lambda \\ &= \frac{1}{t-s} \int_s^t G(s, \lambda) d\lambda,\end{aligned}$$

where  $G(s, \lambda) = \sum_{j \geq 1} \overline{f_j(\lambda)} \langle \mathbf{e}(f), (V_{s,\lambda} - 1)(u, L_j v) \mathbf{e}(0) \rangle$ . Note that the complex valued function  $G(s, \lambda)$  is uniformly continuous in both the variable  $s, \lambda$  on  $[0, \tau]$  and  $G(t, t) = 0$ . So we get

$$\lim_{s \rightarrow t} \langle \mathbf{e}(f), \zeta(s, t, u, v) \rangle = 0.$$

Since  $\zeta(s, t, u, v)$  uniformly bounded in  $s, t$

$$\lim_{s \rightarrow t} \langle \xi, \zeta(s, t, u, v) \rangle = 0, \forall \xi \in \Gamma.$$

2. We have

$$\langle \xi, \gamma(s, t, u, v) \rangle = \frac{1}{t-s} \sum_{j \geq 1} \langle u, L_j v \rangle \int_s^t \overline{\xi_j^{(1)}(\lambda)} d\lambda. \quad (5.18)$$

Since

$$|\sum_{j \geq 1} \langle u, L_j v \rangle \overline{\xi_j^{(1)}(t)}|^2 \leq \|u\|^2 \sum_{j \geq 1} \|L_j v\|^2 \sum_{j \geq 1} |\xi_j^{(1)}(t)|^2 \leq \sum_{j \geq 1} \|L_j v\|^2 \|\xi^{(1)}(t)\|^2,$$

the function  $\sum_{j \geq 1} \langle u, L_j v \rangle \overline{\xi_j^{(1)}}(\cdot) \in L^2$  and hence locally integrable. Thus we get

$$\lim_{s \rightarrow t} \langle \xi, \gamma(s, t, u, v) \rangle = \sum_{j \geq 1} \langle u, L_j v \rangle \overline{\xi_j^{(1)}}(t) \text{ a.e. } t \geq 0.$$

□

**Lemma 5.13.** For  $n \geq 1$ ,  $\underline{t} \in \Sigma_n$  and  $u_k \in \mathbf{h}, v_k \in \mathcal{D} : k = 1, \dots, n, \xi \in \Gamma(L^2(\mathbb{R}_+, \mathbf{k}))$  and  $[s_k, t_k]$ 's are disjoint..

- (i)  $\lim_{\underline{s} \rightarrow \underline{t}} \langle \xi, \prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle = 0$ ,  
where  $M(s_k, t_k, u_k, v_k) = \frac{(V_{s_k, t_k} - 1)}{t_k - s_k}(u_k, v_k) - \langle u_k, G v_k \rangle - \gamma(s_k, t_k, u_k, v_k)$  and  
 $\lim_{\underline{s} \rightarrow \underline{t}}$  means  $s_k \rightarrow t_k$  for each  $k$ .
- (ii)  $\lim_{\underline{s} \rightarrow \underline{t}} \langle \xi, \otimes_{k=1}^n \gamma(s_k, t_k, u_k, v_k) \rangle = \langle \xi^{(n)}(t_1, \dots, t_n), \eta(u_1, v_1) \otimes \dots \otimes \eta(u_n, v_n) \rangle$ .

*Proof.* (i) First note that  $M(s, t, u, v) \mathbf{e}(0) = \zeta(s, t, u, v) + \varsigma(s, t, u, v)$ . So by the above observations  $\{M(s, t, u, v) \mathbf{e}(0)\}$  is uniformly bounded in  $s, t$  and  $\lim_{s \rightarrow t} \langle \mathbf{e}(f), M(s, t, u, v) \mathbf{e}(0) \rangle = 0, \forall f \in L^2(\mathbb{R}_+, \mathbf{k})$ . Since the intervals  $[s_k, t_k]$ 's are disjoint for different  $k$ 's,

$$\langle \mathbf{e}(f), \prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle = \prod_{k=1}^n \langle \mathbf{e}(f|_{[s_k, t_k]}), M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle$$

and thus  $\lim_{\underline{s} \rightarrow \underline{t}} \langle \mathbf{e}(f), \prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0) \rangle = 0$ . By Lemma 5.12, the vector  $\prod_{k=1}^n M(s_k, t_k, u_k, v_k) \mathbf{e}(0)$  is uniformly bounded in  $s_k, t_k$  and hence convergence hold if we replace  $\mathbf{e}(f)$  by any vector  $\xi$  in the Fock Space.

- (ii) It can be proved similarly as in part (iii) of the previous Lemma. □

**Lemma 5.14.** *Let  $\xi \in \Gamma$  be such that*

$$\langle \xi, \zeta \rangle = 0, \quad \forall \zeta \in \mathcal{S}', \quad (5.19)$$

*Then*

- (i)  $\xi^{(0)} = 0$  and  $\xi^{(1)}(t) = 0$  for a.e.  $t \in [0, \tau]$ .
- (ii) For any  $n \geq 0$ ,  $\xi^{(n)}(\underline{t}) = 0$  for a.e.  $\underline{t} \in \Sigma_n : t_i \leq \tau$ .
- (iii) The set  $\mathcal{S}'$  is total in the Fock space  $\Gamma$ .

*Proof.* (i) For any  $s \geq 0$ ,  $V_{s,s} = 1_{\mathbf{h} \otimes \Gamma}$  so in particular (5.19) gives, for any  $u \in \mathbf{h}, v \in \mathcal{D}$

$$0 = \langle \xi, V_{s,s}(u, v) \mathbf{e}(0) \rangle = \langle u, v \rangle \overline{\xi^{(0)}}$$

and hence  $\xi^{(0)} = 0$ .

By (5.19),  $\langle \xi, [V_{s,t} - 1](u, v) \mathbf{e}(0) \rangle = 0$  for any  $0 \leq s < t \leq \tau < \infty, u \in \mathbf{h}, v \in \mathcal{D}$ . Hence for any  $u \in \mathbf{h}, v \in \mathcal{D}$  by Lemma 5.16 (iii) we have

$$\begin{aligned} 0 &= \lim_{s \rightarrow t} \frac{1}{t - s} \langle \xi, [V_{s,t} - 1](u, v) \mathbf{e}(0) \rangle \\ &= \sum_{j \geq 1} \langle u, L_j v \rangle \overline{\xi_j^{(1)}(t)} = \sum_{j \geq 1} \eta_j(u, v) \overline{\xi_j^{(1)}(t)} = \langle \xi^{(1)}(t), \eta(u, v) \rangle \end{aligned}$$

for almost all  $t \in [0, \tau]$ . Since  $\{\eta(u, v) : u \in \mathbf{h}, v \in \mathcal{D}\}$  is total in  $\mathbf{k}$  it follows that  $\xi^{(1)}(t) = 0$  for almost all  $t \leq \tau$ .

(ii) We prove this by induction. The result is already proved for  $n = 0, 1$ . For  $n \geq 2$ , assume as induction hypothesis that for all  $m \leq n - 1$ ,  $\xi^{(m)}(\underline{t}) = 0$ , for a.e.  $\underline{t} \in \Sigma_m : t_k \leq \tau, k = 1, 2, \dots, m$ . We now show that  $\xi^{(n)}(\underline{t}) = 0$ , for a.e.  $\underline{t} \in \Sigma_n : t_k \leq \tau$ .

Let  $0 \leq s_1 < t_1 \leq s_2 < t_2 < \dots < s_n < t_n \leq \tau$  and  $u_k \in \mathbf{h}, v_k \in \mathcal{D} : k = 1, 2, \dots, n$ . By (5.19) and part (i) we have

$$\langle \xi, \prod_{k=1}^n \frac{(V_{s_k, t_k} - 1)}{t_k - s_k} (u_k, v_k) \mathbf{e}(0) \rangle = 0.$$

Thus

$$\begin{aligned} 0 &= \lim_{\underline{s} \rightarrow \underline{t}} \langle \xi, \prod_{k=1}^n \frac{(V_{s_k, t_k} - 1)}{t_k - s_k} (u_k, v_k) \mathbf{e}(0) \rangle \\ &= \lim_{\underline{s} \rightarrow \underline{t}} \langle \xi, \prod_{k=1}^n \{M(s_k, t_k, u_k, v_k) + \langle u_k, G v_k \rangle + \gamma(s_k, t_k, u_k, v_k)\} \mathbf{e}(0) \rangle. \end{aligned} \quad (5.20)$$

Let  $P, Q, R$  and  $P', R'$  be two sets of disjoint partitions of  $\{1, 2, \dots, n\}$  such that  $Q$  and  $R$  are non empty. We write  $|S|$  for the cardinality of set  $S$ . Then by Lemma 5.13 (ii) the right hand side of (5.20) is equal to

$$\begin{aligned} & \sum_{P', R'} \langle \xi^{(|R'|)}(t_{r'_1}, \dots, t_{r'_{|R'|}}), \otimes_{k \in R'} \eta(u_k, v_k) \rangle \prod_{k \in P'} \langle u_k, G v_k \rangle \\ & + \lim_{\underline{s} \rightarrow \underline{t}} \sum_{P, Q, R} \langle \xi, \prod_{k \in P} \langle u_k, G v_k \rangle \prod_{k \in Q} \{M(s_k, t_k, u_k, v_k)\} \prod_{k \in R} \{\gamma(s_k, t_k, u_k, v_k)\} \mathbf{e}(0) \rangle. \end{aligned}$$

Thus by the induction hypothesis,

$$\begin{aligned} 0 &= \langle \xi^{(n)}(t_1, t_2, \dots, t_n), \eta(u_1, v_1) \otimes \dots \otimes \eta(u_n, v_n) \rangle \\ &+ \lim_{\underline{s} \rightarrow \underline{t}} \sum_{P, Q, R} \langle \xi, \prod_{k \in P} \langle u_k, G v_k \rangle \prod_{k \in Q} \{M(s_k, t_k, u_k, v_k)\} \prod_{k \in R} \{\gamma(s_k, t_k, u_k, v_k)\} \mathbf{e}(0) \rangle. \end{aligned} \quad (5.21)$$

We claim that the second term in (5.21) vanishes. To prove the claim, it is enough to show that for any two non empty disjoint subsets  $Q \equiv \{q_1, q_2, \dots, q_{|Q|}\}, R \equiv \{r_1, r_2, \dots, r_{|R|}\}$  of  $\{1, 2, \dots, n\}$ ,

$$\lim_{\underline{s} \rightarrow \underline{t}} \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \mathbf{e}(0) \rangle = 0. \quad (5.22)$$

Writing  $\psi$  for the vector  $\prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \mathbf{e}(0)$ , we have

$$\begin{aligned} & \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \mathbf{e}(0) \rangle \\ &= \langle \xi, \psi \otimes \otimes_{r \in R} \frac{1_{[s_r, t_r]} \eta(u_r, v_r)}{t_r - s_r} \rangle \\ &= \langle \xi, \psi \otimes \otimes_{r \in R} \frac{1_{[s_r, t_r]} \eta(u_r, v_r)}{t_r - s_r} \rangle \\ &= \sum_{l \geq |R|} \langle \xi^{(l)}, \psi^{(l-|R|)} \otimes \otimes_{r \in R} \frac{1_{[s_r, t_r]} \eta(u_r, v_r)}{t_r - s_r} \rangle \\ &= \langle \sum_{l \geq |R|} \langle \xi^{(l)}, \psi^{(l-|R|)} \rangle, \otimes_{r \in R} \frac{1_{[s_r, t_r]} \eta(u_r, v_r)}{t_r - s_r} \rangle. \end{aligned} \quad (5.23)$$

Here  $\langle \psi^{(l-|R|)}, \xi^{(l)} \rangle \in L^2(\mathbb{R}_+, \mathbf{k})^{\otimes |R|}$  is defined as in (2.1) by

$$\begin{aligned} & \langle \langle \psi^{(l-|R|)}, \xi^{(l)} \rangle, \rho^{(|R|)} \rangle = \langle \xi^{(l)}, \psi^{(l-|R|)} \otimes \rho^{(|R|)} \rangle \\ &= \int_{\Sigma_l} \langle \xi^{(l)}(x_1, x_2, \dots, x_l), \\ & \quad \psi^{(l-|R|)}(x_1, x_2, \dots, x_{l-|R|}) \otimes \rho^{(|R|)}(x_{l-|R|+1}, \dots, x_l) \rangle_{\mathbf{k}^{\otimes l}} dx \end{aligned} \quad (5.24)$$

for any  $\rho^{(|R|)} \in L^2(\mathbb{R}_+, \mathbf{k})^{\otimes |R|}$ .

By Lemma 5.13 (i),

$$\lim_{s_q \rightarrow t_q} \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \mathbf{e}(0) \rangle = 0. \quad (5.25)$$

However, we need to prove (5.22) where the limit  $\underline{s} \rightarrow \underline{t}$  has to be in arbitrary order. On the other hand, by (5.23) and (5.24) we get

$$\begin{aligned} & \lim_{s_q \rightarrow t_q} \lim_{s_r \rightarrow t_r} \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \mathbf{e}(0) \rangle \\ &= \lim_{s_q \rightarrow t_q} \lim_{s_r \rightarrow t_r} \langle \sum_{l \geq |R|} \langle \langle \psi^{(l-|R|)}, \xi^{(l)} \rangle \rangle, \otimes_{r \in R} \frac{1_{[s_r, t_r]} \eta(u_r, v_r)}{t_r - s_r} \rangle \\ &= \lim_{s_q \rightarrow t_q} \lim_{s_r \rightarrow t_r} \langle \int_{\Sigma_{|R|}} \langle [\sum_{l \geq |R|} \langle \langle \psi^{(l-|R|)}, \xi^{(l)} \rangle \rangle](x_1, x_2, \dots, x_{|R|}), \\ & \quad \otimes_{r \in R} \frac{1_{[s_r, t_r]}(x_r) \eta(u_r, v_r)}{t_r - s_r} \rangle dx \\ &= \lim_{s_q \rightarrow t_q} \langle \sum_{l \geq |R|} \langle \langle \psi^{(l-|R|)}, \xi^{(l)} \rangle \rangle(t_{r_1}, \dots, t_{r_{|R|}}), \otimes_{r \in R} \eta(u_r, v_r) \rangle, \end{aligned} \quad (5.26)$$

for almost all  $\underline{t} \in \Sigma_{|R|}$ . We fix  $\underline{t} \in \Sigma_{|R|}$  and define families of vectors  $\tilde{\xi}^{(l)} : l \geq 0$  in  $L^2(\mathbb{R}_+, \mathbf{k})^{\otimes l}$  by

$$\begin{aligned} \tilde{\xi}^{(0)} &= \langle \xi^{(|R|)}(t_{r_1}, \dots, t_{r_{|R|}}), \otimes_{r \in R} \eta(u_r, v_r) \rangle \in \mathbb{C} \\ \tilde{\xi}^{(l)}(x_1, x_2, \dots, x_l) &= \langle \langle \otimes_{r \in R} \eta(u_r, v_r), \xi^{(|R|+l)}(x_1, \dots, x_l, t_{r_1}, \dots, t_{r_{|R|}}) \rangle \rangle, \end{aligned}$$

which defines a Fock space vector  $\tilde{\xi}$ . Therefore, from (5.26), we get that

$$\begin{aligned} & \lim_{s_q \rightarrow t_q} \lim_{s_r \rightarrow t_r} \langle \xi, \prod_{q \in Q} \{M(s_q, t_q, u_q, v_q)\} \prod_{r \in R} \{\gamma(s_r, t_r, u_r, v_r)\} \mathbf{e}(0) \rangle = \lim_{s_q \rightarrow t_q} \langle \tilde{\xi}, \psi \rangle \\ &= \lim_{s_q \rightarrow t_q} \langle \tilde{\xi}, [\prod_{q \in Q} M(s_q, t_q, u_q, v_q)] \mathbf{e}(0) \rangle, \end{aligned}$$

which is equal to 0 by Lemma 5.13 (i). Thus from (5.21) we get that

$$\langle \xi^{(n)}(t_1, t_2, \dots, t_n), \eta(u_1, v_1) \otimes \dots \otimes \eta(u_n, v_n) \rangle = 0.$$

Since  $\{\eta(u, v) : u \in \mathbf{h}, v \in \mathcal{D}\}$  is total in  $\mathbf{k}$ , it follows that  $\xi^{(n)}(t_1, t_2, \dots, t_n) = 0$  for almost every  $(t_1, t_2, \dots, t_n) \in \Sigma_n : t_k \leq \tau$ .  $\square$

(iii) Since  $\tau \geq 0$  is arbitrary  $\xi^{(n)} = 0 \in L^2(\mathbb{R}_+, \mathbf{k})^{\otimes n} : n \geq 0$  and hence  $\xi = 0$ . Which proves the totality of  $\mathcal{S}' \subseteq \Gamma$ .

## 5.2 Unitary Equivalence

Here we shall prove the part (ii) of the **Theorem 5.2** that the unitary evolution  $\{U_t\}$  on  $\mathbf{h} \otimes \mathcal{H}$  is unitarily equivalent to the unitary solution  $\{V_t\}$  of HP equation (5.1). To prove this we need the following two results. Let us recall that the subset  $\mathcal{S} = \{\xi = U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega := U_{s_1, t_1}(u_1, v_1) \cdots U_{s_n, t_n}(u_n, v_n)\Omega : \underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n) : 0 \leq s_1 \leq t_1 \leq \dots \leq s_n \leq t_n < \infty, n \geq 1, \underline{u} = \otimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \otimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}\}$  is total in  $\mathcal{H}$  and the subset  $\mathcal{S}' := \{\zeta = V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) := V_{s_1, t_1}(u_1, v_1) \cdots V_{s_n, t_n}(u_n, v_n)\mathbf{e}(0) : \underline{u} = \otimes_{i=1}^n u_i \in \mathbf{h}^{\otimes n}, \underline{v} = \otimes_{i=1}^n v_i \in \mathcal{D}^{\otimes n}, \underline{s} = (s_1, s_2, \dots, s_n), \underline{t} = (t_1, t_2, \dots, t_n)\}$  is total in  $\Gamma$ .

**Lemma 5.15.** *Let  $U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega \in \mathcal{S}$ .*

*Then there exist an integer  $m \geq 1$ ,  $\underline{a} = (a_1, a_2, \dots, a_m), \underline{b} = (b_1, b_2, \dots, b_m) : 0 \leq a_1 \leq b_1 \leq \dots \leq a_m \leq b_m < \infty$ , partition  $R_1 \cup R_2 \cup R_3 = \{1, \dots, m\}$  with  $|R_i| = m_i$ , family of vectors  $x_{k_l}, g_{k_l} \in \mathbf{h}$  and  $y_{k_l}, h_{k_l} \in \mathcal{D} : l \in R_1 \cup R_2, i \in R_2 \cup R_3$  such that*

$$U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v}) = \sum_{\underline{k}} \prod_{l \in R_1 \cup R_2} U_{a_l, b_l}(x_{k_l}, y_{k_l}) \quad (5.27)$$

$$U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w}) = \sum_{\underline{k}} \prod_{l \in R_2 \cup R_3} U_{a_l, b_l}(g_{k_l}, h_{k_l}). \quad (5.28)$$

*Proof.* It follows from the evolution hypothesis of the family of unitary operators  $\{U_{s,t}\}$  as for  $r \in [s, t]$  and orthonormal basis  $\{f_j\} \subseteq \mathcal{D}$  of  $\mathbf{h}$  we can write  $U_{s,t}(u, v) = \sum_{j \geq 1} U_{s,r}(u, f_j)U_{r,t}(f_j, v)$ .  $\square$

**Remark 5.16.** *Since the family of unitaries  $\{V_{s,t}\}$  on  $\mathbf{h} \otimes \Gamma$  enjoy all the properties satisfy by family of unitaries  $\{U_{s,t}\}$  on  $\mathbf{h} \otimes \mathcal{H}$  the above Lemma also hold if we replace  $U_{s,t}$  by  $V_{s,t}$ .*

**Lemma 5.17.** *For  $U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega \in \mathcal{S}$ .*

$$\langle U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega \rangle = \langle V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0), V_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\mathbf{e}(0) \rangle. \quad (5.29)$$



*Proof.* We have by previous Lemma and assumption **A**

$$\begin{aligned}
& \langle U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega, U_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\Omega \rangle \\
&= \sum_{\underline{k}} \prod_{l \in R_1} \langle U_{b_l - a_l}(x_{k_l}, y_{k_l})\Omega, \Omega \rangle \prod_{l \in R_2} \langle U_{b_l - a_l}(x_{k_l}, y_{k_l})\Omega, U_{b_l - a_l}(g_{k_l}, h_{k_l})\Omega \rangle \\
& \quad \prod_{l \in R_3} \langle \Omega, U_{b_l - a_l}(g_{k_l}, h_{k_l})\Omega \rangle \\
&= \sum_{\underline{k}} \prod_{l \in R_1} \langle T_{b_l - a_l} y_{k_l}, x_{k_l} \rangle \prod_{l \in R_2} \langle g_{k_l}, Z_{b_l - a_l}(|h_{k_l} \rangle \langle y_{k_l}|) x_{k_l} \rangle \prod_{l \in R_3} \langle g_{k_l}, T_{b_l - a_l} h_{k_l} \rangle \\
&= \sum_{\underline{k}} \prod_{l \in R_1} \langle V_{b_l - a_l}(x_{k_l}, y_{k_l})\mathbf{e}(0), \mathbf{e}(0) \rangle \prod_{l \in R_2} \langle V_{b_l - a_l}(x_{k_l}, y_{k_l})\mathbf{e}(0), V_{b_l - a_l}(g_{k_l}, h_{k_l})\mathbf{e}(0) \rangle \\
& \quad \prod_{l \in R_3} \langle \mathbf{e}(0), V_{b_l - a_l}(g_{k_l}, h_{k_l})\mathbf{e}(0) \rangle.
\end{aligned}$$

Now by Remark (5.16), the above quantity is equal to  $\langle V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0), V_{\underline{s}', \underline{t}'}(\underline{p}, \underline{w})\mathbf{e}(0) \rangle$ .  $\square$

**Proof of the part (ii) of Theorem 5.2 :**

We need to construct a unitary operator  $\tilde{\Xi} : \mathbf{h} \otimes \mathcal{H} \rightarrow \mathbf{h} \otimes \Gamma$  such that

$$U_t = \tilde{\Xi}^* V_t \tilde{\Xi}, \quad \forall \quad t \geq 0. \quad (5.30)$$

Let us define a map  $\Xi : \mathcal{H} \rightarrow \Gamma$  by setting, for any  $\xi = U_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\Omega \in \mathcal{S}$ ,  $\Xi\xi := V_{\underline{s}, \underline{t}}(\underline{u}, \underline{v})\mathbf{e}(0) \in \mathcal{S}'$  and then extending linearly. So by definition and totality of  $\mathcal{S}'$ , range of  $\Xi$  is dense in  $\Gamma$ . To see that  $\Xi$  is a unitary operator from  $\mathcal{H}$  to  $\Gamma$  it is enough to note from Lemma 5.17 that

$$\langle \Xi\xi, \Xi\xi' \rangle = \langle \xi, \xi' \rangle, \quad \forall \quad \xi, \xi' \in \mathcal{S}. \quad (5.31)$$

For the conclusion it suffices to set  $\tilde{\Xi} = 1_{\mathbf{h}} \otimes \Xi$ .  $\square$

**Remark 5.18.** *The assumption **C** is ruling out the presence of conservation (Poisson) terms in the associated HP equation as the representation  $\pi$ , we obtained, is trivial (see Remark 4.2). Without this assumption **C**, the problem is not yet settled. In the absence of assumption **C** the representation  $\pi$  shall be non trivial which in general will give rise to a unitary (different from identity) operator  $W$  on  $\mathbf{h} \otimes \mathbf{k}$  and associated HP equation (5.1) will contain conservation terms with coefficients  $\{L_\nu^\mu\}$  described as in (1.2).*

**Remark 5.19.** *The Hypothesis **E2**, i.e. there exists  $\mathcal{D}$ , core for  $G$  such that  $\mathcal{D} \subseteq \mathcal{D}(L_j^*)$  for every  $j \geq 1$ , is a strong assumption. But this is necessary one*

in order that quantum stochastic differential equation for  $V_t$  makes sense. Only way one can do away with this assumption is to abandon the quantum stochastic differential equation for  $V_t$  and just deal with  $V_t$  as a left cocycle described by the associated four semigroups [9]. This programme is not yet complete.

**Remark 5.20.** *The Hypothesis **E3**, i.e. for any  $v \in \mathcal{D}, \sum_{j \geq 1} \|GL_j v\|^2 < \infty$ . This holds trivially when  $[G, L_j] = 0$ . Condition  $[G, L_j] = 0$ , in particular holds for classical Brownian motion on  $\mathbb{R}^n$  and for Casimir operator  $G$  on Lie algebra of a locally compact Lie group  $\mathcal{G}$  with  $L_j = X_j$  represented on the Hilbert space  $\mathfrak{h} = L^2(\mathcal{G})$ , where  $\{X_j\}_{j=1}^n$  a basis for the Lie algebra. The commutator  $[G, L_j]$  also vanish in case of Quantum Brownian motion on non-commutative Torus, Quantum Heisenberg manifold and Quantum Plane [4].*

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